第36卷 第2期 复旦学报(自然科学版) Vol.36 No.2
1997年4月 Journal of Fudan University (Natural Science) APR. 1997

二次参数曲面的可展分类

曹 沅 华宣积
(数 学 系)

提 要 利用相对仿射不变量讨论了二次参数曲面的可展性. 指出二次参数曲面成为可展曲面的情况只有两种:一次曲面,即平面;二次曲面中的抛物柱面或二次锥面.

关键词 可展曲面;高斯曲率;仿射变换;不变量
中图法分类号 O 186.13

1 引理和引言

在计算几何中,曾经用相对仿射不变量来研究曲线曲面的凸性和连续性[1~7],本文则用其对二次参数曲面的可展性作一探讨. 一般地说,二次参数曲面对应于四次代数曲面,我们希望了解在可展的限制下它们的类型.

已知正则曲面

$$\vec{r}(u,v)=(x(u,v),y(u,v),z(u,v))\qquad(u,v)\in\mathscr{D}\subset R^2\tag{1}$$

记 $R=(\vec{r}_u,\vec{r}_v,\vec{r}_{uu}),\ S=(\vec{r}_u,\vec{r}_v,\vec{r}_{uv}),\ T=(\vec{r}_u,\vec{r}_v,\vec{r}_{vv}),$

其中 $\vec{r}_u=\dfrac{\partial\vec{r}}{\partial u},\vec{r}_v=\dfrac{\partial\vec{r}}{\partial v},\ \vec{r}_{uu}=\dfrac{\partial^2\vec{r}}{\partial u^2},\vec{r}_{uv}=\dfrac{\partial^2\vec{r}}{\partial u\partial v},\vec{r}_{vv}=\dfrac{\partial^2\vec{r}}{\partial v^2},$

则曲面(1)的高斯曲率 K 与 $RT\text{-}S^2$ 有相同的符号.

在讨论中,我们要用到如下三个引理.

引理 1 在仿射变换

$$\bar{\vec{r}}=A\vec{r}+\vec{r}_0,\qquad |A|\neq 0\tag{2}$$

下,R,S,T 是关于仿射变换(2)的权 1 的相对仿射不变量. 从而,$RT\text{-}S^2$ 是关于仿射变换(2)的权 2 的相对仿射不变量.

引理 2 在参数变换

$$\begin{cases}u=u(\bar{u},\bar{v}),\\ v=v(\bar{u},\bar{v}),\end{cases}\qquad\left|\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})}\right|\neq 0\tag{3}$$

下,$EG\text{-}F^2$ 是关于(3)式的权 2 的相对不变量,$RT\text{-}S^2$ 是关于(3)式的权 4 的相对不变量,

收稿日期:1995-06-23
第一作者曹 沅,副教授;复旦大学数学系,上海 200433

其中 $E=\vec{r}_u\cdot\vec{r}_u$, $F=\vec{r}_u\cdot\vec{r}_v$, $G=\vec{r}_v\cdot\vec{r}_v$.

以上两个引理的证明见参考文献[3].

由引理1和引理2,得

引理3 sign(K)是仿射变换(2)和参数变换(3)下的不变量.

2 二次参数曲面的可展分类

考虑如下的二次参数曲面

$$\vec{r}(u,v)=\vec{r}_1u^2+\vec{r}_2uv+\vec{r}_3v^2+\vec{r}_4u+\vec{r}_5v+\vec{r}_6 \qquad (u,v)\in\mathscr{D}\subset R^2, \tag{4}$$

其分量形式是

$$\begin{cases}x=a_1u^2+a_2uv+a_3v^2+a_4u+a_5v+a_6\\ y=b_1u^2+b_2uv+b_3v^2+b_4u+b_5v+b_6 \qquad (u,v)\in\mathscr{D}\subset R^2,\\ z=c_1u^2+c_2uv+c_3v^2+c_4u+c_5v+c_6\end{cases} \tag{4'}$$

记矩阵

$$m_{ijk}=(\vec{r}_i,\vec{r}_j,\vec{r}_k), \tag{5}$$

和行列式

$$d_{ijk}=|(\vec{r}_i,\vec{r}_i,\vec{r}_k)|, \tag{6}$$

由于u,v的对称性,我们可将问题分解如下:

(1) $d_{123}\neq0$;

(2) $d_{123}=0,d_{125}\neq0$,

$d_{123}=0,d_{324}\neq0$;

(3) $d_{123}=0,d_{125}=0,d_{324}=0,d_{124}\neq0$,

$d_{123}=0,d_{125}=0,d_{324}=0,d_{325}\neq0$;

(4) $d_{123}=0,d_{125}=0,d_{324}=0,d_{124}=0,d_{325}=0,d_{134}\neq0$,

$d_{123}=0,d_{125}=0,d_{324}=0,d_{124}=0,d_{325}=0,d_{135}\neq0$;

(5) $d_{123}=0,d_{125}=0,d_{324}=0,d_{124}=0,d_{325}=0,d_{134}=0,d_{135}=0,d_{345}\neq0$,

$d_{123}=0,d_{125}=0,d_{324}=0,d_{124}=0,d_{325}=0,d_{134}=0,d_{135}=0,d_{145}\neq0$;

(6) $d_{123}=0,d_{125}=0,d_{324}=0,d_{124}=0,d_{325}=0$,

$d_{134}=0,d_{135}=0,d_{345}=0,d_{145}=0,d_{245}\neq0$;

(7) $d_{ijk}=0$, $1\leqslant i,j,k\leqslant5$, $i\neq j$, $j\neq k$, $k\neq i$.

不考虑向量$\vec{r}_6$是因为它对曲面的可展性没有影响.

现分类讨论

(Ⅰ) 因为$d_{123}\neq0$,作仿射变换

$$\bar{\vec{r}}=m_{123}^{-1}\vec{r}, \tag{7}$$

$$\begin{cases}\bar{x} = u^2 + \bar{a}_4 u + \bar{a}_5 v, \\ \bar{y} = uv + \bar{b}_4 u + \bar{b}_5 v \qquad (u,v) \in \mathscr{D} \subset R^2, \\ \bar{z} = v^2 + \bar{c}_4 u + \bar{c}_5 v,\end{cases} \tag{7'}$$

其中

$$\bar{a}_4 = \frac{d_{234}}{d_{123}},\ \bar{a}_5 = \frac{d_{235}}{d_{123}},\ \bar{b}_4 = -\frac{d_{134}}{d_{123}},\ \bar{b}_5 = -\frac{d_{135}}{d_{123}},\bar{c}_4 = -\frac{d_{124}}{d_{123}},\ \bar{c}_5 = \frac{d_{125}}{d_{123}}, \tag{8}$$

再作参数变换

$$\begin{cases}u = \tilde{u} - \bar{b}_5, \\ v = \tilde{v} - \bar{b}_4,\end{cases} \tag{9}$$

删去常数项，并记

$$\tilde{a}_4 = \bar{a}_4 - 2\bar{b}_5, \tilde{a}_5 = \bar{a}_5, \tilde{c}_4 = \bar{c}_4, \tilde{c}_5 = \bar{c}_5 - 2\bar{b}_4,$$

则有

$$\begin{cases}\bar{x} = \tilde{u}^2 + \tilde{a}_4\tilde{u} + \tilde{a}_5\tilde{v}, \\ \bar{y} = \tilde{u}\tilde{v} \qquad (\tilde{u},\tilde{v}) \in \widetilde{\mathscr{D}} \subset R^2; \\ \bar{z} = \tilde{v}^2 + \tilde{c}_4\tilde{u} + \tilde{c}_5\tilde{v},\end{cases} \tag{10}$$

计算

$$\begin{cases}R = 2(2\tilde{v}^2 - \tilde{c}_4\tilde{u} + \tilde{c}_5\tilde{v}), \\ S = -4\tilde{u}\tilde{v} - 2\tilde{c}_5\tilde{u} - 2\tilde{a}_4\tilde{v} + \tilde{a}_5\tilde{c}_4 - \tilde{a}_4\tilde{c}_5, \\ T = 2(2\tilde{u}^2 + \tilde{a}_4\tilde{u} - \tilde{a}_5\tilde{v}),\end{cases}$$

$$\begin{aligned}RT - S^2 = &-8\tilde{c}_4\tilde{u}^3 - 8\tilde{c}_5\tilde{u}^2\tilde{v} - 8\tilde{a}_4\tilde{u}\tilde{v}^2 - 8\tilde{a}_5\tilde{v}^3 - \\ &4(\tilde{a}_4\tilde{c}_5 + \tilde{c}_5^2)\tilde{u}^2 + 12(\tilde{a}_5\tilde{c}_4 - \tilde{a}_4\tilde{c}_5)\tilde{u}\tilde{v} - 4(\tilde{a}_5\tilde{c}_5 + \tilde{a}_4^2)\tilde{v}^2 + \\ &4\tilde{c}_5(\tilde{a}_5\tilde{c}_4 - \tilde{a}_4\tilde{c}_5)\tilde{u} + 4\tilde{a}_4(\tilde{a}_5\tilde{c}_4 - \tilde{a}_4\tilde{c}_5)\tilde{v} - (\tilde{a}_5\tilde{c}_4 - \tilde{a}_4\tilde{c}_5)^2,\end{aligned} \tag{11}$$

由此可得 $RT-S^2=0$ 的充分必要条件是

$$\tilde{a}_4 = 0,\ \tilde{a}_5 = 0,\ \tilde{c}_4 = 0,\ \tilde{c}_5 = 0. \tag{12}$$

即

$$d_{234} + 2d_{135} = 0,\ d_{235} = 0,\ d_{124} = 0,\ d_{125} + 2d_{134} = 0. \tag{13}$$

再从(10)和(12)两式可得

$$\bar{y}^2 = \bar{x}\bar{z}, \tag{14}$$

因此在(I)的条件下，可展曲面为二次锥面.

(II)　作仿射变换

$$\bar{\vec{r}} = m_{125}^{-1}\vec{r}, \tag{15}$$

$$\begin{cases}\bar{x} = u^2 + \bar{a}_3 v^2 + \bar{a}_4 u, \\ \bar{y} = uv + \bar{b}_3 v^2 + \bar{b}_4 u, \\ \bar{z} = \bar{c}_3 v^2 + \bar{c}_4 u + v,\end{cases} \tag{15'}$$

其中

$$\bar{a}_3=-\frac{d_{235}}{d_{125}},\ \bar{a}_4=-\frac{d_{245}}{d_{125}},\ \bar{b}_3=\frac{d_{135}}{d_{125}},\ \bar{b}_4=\frac{d_{145}}{d_{125}},\ \bar{c}_3=\frac{d_{123}}{d_{125}}=0,\ \bar{c}_4=\frac{d_{124}}{d_{125}},\tag{16}$$

作参数变换

$$\begin{cases}u=\tilde{u}+2\bar{b}_3\bar{b}_4,\\ v=\tilde{v}-\bar{b}_4,\end{cases}\tag{17}$$

删去常数项,并记

$$\tilde{a}_3=\bar{a}_3,\ \tilde{a}_4=4\bar{b}_3\bar{b}_4+\bar{a}_4,\ \tilde{a}_5=-2\bar{a}_3\bar{b}_4,\ \tilde{b}_3=\bar{b}_3,\ \tilde{c}_4=\bar{c}_4,$$

有

$$\begin{cases}\bar{x}=\tilde{u}^2+\tilde{a}_3\tilde{v}^2+\tilde{a}_4\tilde{u}+\tilde{a}_5\tilde{v},\\ \bar{y}=\tilde{u}\tilde{v}+\tilde{b}_3\tilde{v}^2,\\ \bar{z}=\tilde{c}_4\tilde{u}+\tilde{v}.\end{cases}\tag{18}$$

计算

$$\begin{cases}R=2(-\tilde{c}_4\tilde{u}+(1-2\tilde{b}_3\tilde{c}_4)\tilde{v}),\\ S=-2\tilde{u}+2\tilde{a}_3\tilde{c}_4\tilde{v}+\tilde{c}_4\tilde{a}_5-\tilde{a}_4,\\ T=2((-\tilde{a}_3\tilde{c}_4-2\tilde{b}_3)\tilde{u}+\tilde{a}_3\tilde{v}-\tilde{b}_3(\tilde{a}_4-\tilde{a}_5\tilde{c}_4));\end{cases}$$

$$\begin{aligned}RT-S^2=&4(\tilde{c}_4(\tilde{a}_3\tilde{c}_4+2\tilde{b}_3)-1)\tilde{u}^2+8\tilde{b}_3(\tilde{c}_4(\tilde{a}_3\tilde{c}_4+2\tilde{b}_3)-1)\tilde{u}\tilde{v}-\\&4\tilde{a}_3(\tilde{c}_4(\tilde{a}_3\tilde{c}_4+2\tilde{b}_3)-1)\tilde{v}^2+4(\tilde{c}_4\tilde{a}_5-\tilde{a}_4)(1-\tilde{c}_4\tilde{b}_3)\tilde{u}+\\&4(\tilde{c}_4\tilde{a}_5-\tilde{a}_4)(\tilde{b}_3(1-2\tilde{b}_3\tilde{c}_4)-\tilde{a}_3\tilde{c}_4)\tilde{v}-(\tilde{c}_4\tilde{a}_5-\tilde{a}_4)^2,\end{aligned}\tag{19}$$

由此 $RT-S^2=0$ 的充分必要条件为

$$\begin{cases}\tilde{c}_4\tilde{a}_5-\tilde{a}_4=0,\\ \tilde{a}_3\tilde{c}_4^2+2\tilde{b}_3\tilde{c}_4=1,\end{cases}\tag{20}$$

即

$$\begin{cases}2d_{124}d_{235}d_{145}-4d_{135}d_{145}d_{125}+d_{245}d_{125}^2=0,\\ -d_{235}d_{124}^2+2d_{135}d_{124}d_{125}=d_{125}^2.\end{cases}\tag{20'}$$

由(18)与(20)两式,我们有

$$\tilde{c}_4^2\bar{x}+2\tilde{c}_4\bar{y}=\bar{z}^2+\tilde{c}_4^2\tilde{a}_5\bar{z},\tag{21}$$

故可展曲面为抛物柱面.

对(Ⅱ)中的后一种情况,只要将足标 1 与 3,4 与 5 对换即可得到.

对(Ⅲ),(Ⅳ),(Ⅴ),作完全类似的工作可知,可展曲面为抛物柱面.

(Ⅵ) 作仿射变换

$$\bar{\vec{r}}=m_{245}^{-1}\vec{r},\tag{22}$$

$$\begin{cases}\bar{x}=\bar{a}_1u^2+uv+\bar{a}_3v^2,\\ \bar{y}=\bar{b}_1u^2+\bar{b}_3v^2+u,\\ \bar{z}=\bar{c}_1u^2+\bar{c}_3v^2+v,\end{cases}\tag{22'}$$

其中

$$\begin{cases}\bar{a}_1 = \dfrac{d_{145}}{d_{245}} = 0, \bar{a}_3 = \dfrac{d_{345}}{d_{245}} = 0, \bar{b}_1 = \dfrac{d_{215}}{d_{245}} = 0, \\ \bar{b}_3 = \dfrac{d_{235}}{d_{245}} = 0, \bar{c}_1 = \dfrac{d_{241}}{d_{245}} = 0, \bar{c}_3 = \dfrac{d_{243}}{d_{245}} = 0. \end{cases} \tag{23}$$

由此有

$$\bar{x} = \overline{y}\overline{z}. \tag{24}$$

这是马鞍面，不可展，在所有处有 $RT-S^2=-1$

（Ⅶ） 由条件可知秩 $\mathrm{Ran}(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4, \vec{r}_5) \leqslant 2$，

存在不全为零的 α, β, γ，使成立

$$\alpha x + \beta y + \gamma z = 0. \tag{25}$$

这是平面，当然是可展的.

综上所述，可得

定理 二次参数曲面(4)可由其权 1 的相对仿射不变量 d_{ijk} 分为 7 种类型. 其中一类为马鞍面，一类为平面，另五类如果可展，则为二次锥面或抛物柱面.

推论 可展的二次参数曲面(4)不可能为切线面或其他柱面和锥面.

参 考 文 献

1 苏步青. 关于三次参数样条曲线的一个定理. 应用数学学报，1977，**1**(1)：49～54

2 苏步青. 论 Bézier 曲线的仿射不变量. 计算数学，1980，**2**(4)：289～298

3 苏步青. 仿射微分几何. 北京：科学出版社，1982

4 华宣积，邝志全. Bézier 曲面的凸性定理. 浙江大学学报，"计算几何"讨论会论文集，1982

5 Chang G, Davis PJ. The convexity of Bernstein polynomials over triangles. *J Approx Theory*, 1984, **40**: 11～28

6 Chang Gengzhe, Feng Yuyu. A new proof for the convexity of the Bernstein-Bézier surfaces over triangles. *Chin Ann of Math* (*Ser* B), **6**(2): 171～175

7 Cao Yuan, Hua Xuanji. The convexity of quadratic parametric triangular Bernstein-Bézier surface. *Computer Aidcd Geometric Design*, 1991, **8**(1): 1～6

Developable classification of quadratic parametric surface

Cao Yuan, Hua Xuanji
(*Department of Mathematics*)

Abstract This paper discusses the developable classification of quadratic parametric surfaces by relative affine invariants and points out that only two categories are developable surface ; i. e. ,plane,and quadratic cone or parabolic cylinder pertaining to quadratic surfaces.

Keywords developable surface; Gaussian curvature; invariant; affine transformation