《微分几何》第 143 页习题解答

1. 平面上取极坐标系时, 第一基本形式为 $\mathrm{d}s^2=\mathrm{d}\rho^2+\rho^2\mathrm{d}\theta^2$. 试计算第二类克氏符号 Γ_{ij}^k.

解： 由假设, $E(\rho,\theta)=1,\ F(\rho,\theta)=0,\ G(\rho,\theta)=\rho^2$, 按正交网时第二类克氏符号的计算公式, 可直接计算得到

$$\begin{aligned}&\Gamma_{11}^1=\frac{E_\rho}{2E}=0,\quad \Gamma_{11}^2=-\frac{E_\theta}{2G}=0,\quad \Gamma_{12}^1=\Gamma_{21}^1=\frac{E_\theta}{2E}=0,\\ &\Gamma_{22}^1=-\frac{G_\rho}{2E}=\rho,\quad \Gamma_{22}^2=\frac{G_\theta}{2G}=0,\quad \Gamma_{12}^2=\gamma_{21}^2=\frac{G_\rho}{2G}=\frac{1}{\rho}.\end{aligned}$$

2. 证明高斯曲率 $K=\det(\mu_i^j)$.

证明： 首先, 容易验证

$$\begin{pmatrix}g^{11}&g^{12}\\g^{21}&g^{22}\end{pmatrix}=\frac{1}{g}\begin{pmatrix}g_{22}&-g_{12}\\-g_{21}&g_{11}\end{pmatrix}.$$

将其代入 μ_i^j 的计算公式: $\mu_i^j=-\sum\limits_k g^{jk}L_{ik}$, 直接计算得到

$$\begin{pmatrix}\mu_1^1&\mu_1^2\\\mu_2^1&\mu_2^2\end{pmatrix}=\frac{1}{EG-F^2}\begin{pmatrix}MF-LG&LF-ME\\NF-MG&MF-NE\end{pmatrix}. \tag{2.1}$$

于是

$$\det(\mu_i^j)=\frac{LN-M^2}{EG-F^2}=K.$$

3. 证明平均曲率 $H=-\frac{1}{2}\mathrm{trace}(\mu_i^j)$.

证明： 从 (2.1) 式立即可得到结论.

4. 求证:

$$\begin{aligned}R_{mijk}=&\frac{1}{2}\left(\frac{\partial^2 g_{mj}}{\partial u^i\partial u^k}+\frac{\partial^2 g_{ik}}{\partial u^m\partial u^j}-\frac{\partial^2 g_{mk}}{\partial u^i\partial u^j}-\frac{\partial^2 g_{ij}}{\partial u^m\partial u^k}\right)+\\&\sum_p\left(\Gamma_{ik}^p\cdot[mj,p]-\Gamma_{ij}^p\cdot[mk,p]\right).\end{aligned}$$

证明：回忆第一、第二类黎曼曲率张量的定义:

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ij}}{\partial u^k} - \frac{\partial \Gamma^l_{ik}}{\partial u^j} + \sum_p (\Gamma^p_{ij}\Gamma^l_{pk} - \Gamma^p_{ik}\Gamma^l_{pj}),$$

$$R_{mijk} = \sum_l g_{ml} R^l_{ijk} = \sum_l g_{ml}\left[\frac{\partial \Gamma^l_{ij}}{\partial u^k} - \frac{\partial \Gamma^l_{ik}}{\partial u^j} + \sum_p (\Gamma^p_{ij}\Gamma^l_{pk} - \Gamma^p_{ik}\Gamma^l_{pj})\right]. \tag{4.1}$$

由于 $[ij,m] = \sum\limits_l \Gamma^l_{ij} g_{lm}$, 所以

$$\frac{\partial [ij,m]}{\partial u^k} = \sum_l \frac{\partial \Gamma^l_{ij}}{\partial u^k} g_{ml} + \sum_l \frac{\partial g_{lm}}{\partial u^k}\Gamma^l_{ij},$$

即

$$\sum_l g_{ml}\frac{\partial \Gamma^l_{ij}}{\partial u^k} = \frac{\partial [ij,m]}{\partial u^k} - \sum_l \frac{\partial g_{lm}}{\partial u^k}\Gamma^l_{ij}.$$

代入 (4.1) 式得

$$\begin{aligned} R_{mijk} =& \frac{\partial [ij,m]}{\partial u^k} - \frac{\partial [ik,m]}{\partial u^j} - \sum \frac{\partial g_{lm}}{\partial u^k}\Gamma^l_{ij} + \sum_l \frac{\partial g_{ml}}{\partial u^j}\Gamma^l_{ik} \\ &+ \sum_p [pk,m]\Gamma^p_{ij} - \sum_p [pj,m]\Gamma^p_{ik}. \end{aligned} \tag{4.2}$$

由于

$$[ij,m] = \frac{1}{2}(\frac{\partial g_{im}}{\partial u^j} + \frac{\partial g_{jm}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^m}),$$

$$[ik,m] = \frac{1}{2}(\frac{\partial g_{im}}{\partial u^k} + \frac{\partial g_{km}}{\partial u^i} - \frac{\partial g_{ik}}{\partial u^m}),$$

$$\frac{\partial g_{ml}}{\partial u^k} = [ik,m] + [mk,l],$$

$$\frac{\partial g_{ml}}{\partial u^j} = [lj,m] + [mj,l],$$

代入 (4.2) 式得

$$
\begin{aligned}
R_{mijk} =&\frac{1}{2}\frac{\partial}{\partial u^k}\left(\frac{\partial g_{im}}{\partial u^j}+\frac{\partial g_{jm}}{\partial u^i}-\frac{\partial g_{ij}}{\partial u^m}\right)-\frac{1}{2}\frac{\partial}{\partial u^j}\left(\frac{\partial g_{im}}{\partial u^k}+\frac{\partial g_{km}}{\partial u^i}-\frac{\partial g_{ik}}{\partial u^m}\right)\\
&-([lk,m]+[mk,l])\Gamma^l_{ij}+([lj,m]+[mj,l])\Gamma^l_{ik}\\
&+[pk,m]\Gamma^p_{ij}-[p_j,m]\Gamma^p_{ik}\\
=&\frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial u^i\partial u^k}+\frac{\partial^2 g_{ik}}{\partial u^j\partial u^k}-\frac{\partial^2 g_{ij}}{\partial u^m\partial u^k}-\frac{\partial^2 g_{km}}{\partial u^i\partial u^j}\right)\\
&+\sum_l[lj,m]\Gamma^l_{ik}+\sum_l[mj,l]\Gamma^l_{ik}+\sum_p[pk,m]\Gamma^p_{ij}\\
&-\sum_p[pj,m]\Gamma^p_{ik}-\sum_l[lk,m]\Gamma^l_{ij}-\sum_l[mk,l]\Gamma^l_{ij}\\
=&\frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial u^i\partial u^k}+\frac{\partial^2 g_{im}}{\partial u^j\partial u^k}-\frac{\partial^2 g_{ij}}{\partial u^m\partial u^k}-\frac{\partial^2 g_{km}}{\partial u^i\partial u^j}\right)\\
&+\sum_p[mj,p]\Gamma^p_{ik}-\sum_p[mk,p]\Gamma^p_{ij}.
\end{aligned}
$$

5. 对于 $\mathbb{R}^3$ 中的空间曲面来说, $R^l_{ijk}=-K(\delta^l_j g_{ik}-\delta^l_k g_{ij})$, 其中 K 是曲面的 Gauss 曲率.

证明：对于 $\mathbb{R}^3$ 中的曲面来说, R_{mijk} 中的本质分量只有一个, 即

$$R_{1212}=-K(g_{11}g_{22}-g_{12}g_{21}).$$

因此可以一般地表示为

$$R_{mijk}=-K(g_{mj}g_{ik}-g_{mk}g_{ij}).$$

于是

$$
\begin{aligned}
R^l_{ijk}&=\sum_m g^{ml}R_{mijk}\\
&=\sum_m g^{ml}[-K(g_{mj}g_{ik}-g_{mk}g_{ij})]\\
&=-K\left(\sum_m g^{ml}g_{mj}g_{ik}-\sum_m g^{ml}g_{mk}g_{ij}\right)\\
&=-K(\delta^l_j g_{ik}-\delta^l_k g_{ij}).
\end{aligned}
$$

6. 证明以下公式:

(1) $K=\frac{1}{E}[(\Gamma_{11}^2)_v-(\Gamma_{12}^2)_u+\Gamma_{11}^1\Gamma_{12}^2+\Gamma_{11}^2\Gamma_{22}^2-\Gamma_{12}^1\Gamma_{11}^2-(\Gamma_{12}^2)^2]$.

(2) $K=\frac{1}{\sqrt{EG-F^2}}\left[\frac{\partial}{\partial v}\left(\frac{\sqrt{EG-F^2}}{E}\Gamma_{11}^2\right)-\frac{\partial}{\partial u}\left(\frac{\sqrt{EG-F^2}}{E}\Gamma_{12}^2\right)\right]$.

(3) $K=\frac{1}{\sqrt{EG-F^2}}\left[\frac{\partial}{\partial u}\left(\frac{\sqrt{EG-F^2}}{G}\Gamma_{22}^1\right)-\frac{\partial}{\partial v}\left(\frac{\sqrt{EG-F^2}}{G}\Gamma_{12}^1\right)\right]$.

(4) 对于曲面上的等温坐标网有 $\mathrm{d}s^2=\lambda^2(\mathrm{d}u^2+\mathrm{d}v^2)$, 求证:

$$K=-\frac{1}{\lambda^2}\left[(\ln\lambda)_{uu}+(\ln\lambda)_{vv}\right].$$

(5) 对于曲面上的半测地坐标网有 $\mathrm{d}s^2=\mathrm{d}u^2+G\mathrm{d}v^2$, 求证:

$$K=-\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2}.$$

证明: (1) 由高斯公式及第一、第二类黎曼曲率张量的定义式, 易见

$$\frac{\partial\Gamma_{ij}^l}{\partial u^k}-\frac{\partial\Gamma_{ik}^p}{\partial u^j}+\sum_p\left(\Gamma_{ij}^p\Gamma_{pk}^l-\Gamma_{ik}^p\Gamma_{pj}^l\right)=\sum_m g^{ml}\left[L_{ij}L_{mk}-L_{ik}L_{mj}\right].$$

在上式中取 $k=2,\ j=1,\ l=2,\ i=1$, 得到

$$\begin{aligned}&(\Gamma_{11}^2)_v-(\Gamma_{12}^2)_u+\Gamma_{11}^1\Gamma_{12}^2+\Gamma_{11}^2\Gamma_{22}^2-\Gamma_{12}^1\Gamma_{11}^2-(\Gamma_{12}^2)^2\\&\quad=\sum_m g^{m2}[L_{11}L_{2m}-L_{12}L_{1m}].\end{aligned}$$

容易验证上式右端关于 m 求和的结果是 EK, 从而证明了结论(1).

(2) 由于 $K=-\frac{R_{1212}}{g}$, 所以

$$R_{121}^2=g^{22}R_{2121}=g^{22}R_{1212}=-g^{22}gK=-g_{11}K.$$

按定义还有

$$R_{121}^2=\frac{\partial\Gamma_{12}^2}{\partial u^1}-\frac{\partial\Gamma_{11}^2}{\partial u^2}+\Gamma_{12}^2\Gamma_{21}^2-\Gamma_{11}^2\Gamma_{22}^2.$$

因此

$$\begin{aligned}K&=-\frac{1}{g}[(\Gamma_{12}^2)_u-(\Gamma_{11}^2)_v+\Gamma_{12}^1\Gamma_{11}^2+\Gamma_{12}^2\Gamma_{21}^2-\Gamma_{11}^1\Gamma_{12}^2-\Gamma_{11}^2\Gamma_{22}^2]\\&=\frac{1}{g}[(\Gamma_{11}^2)_v-(\Gamma_{12}^2)_u+\Gamma_{11}^2(\Gamma_{22}^2+\Gamma_{12}^1)-\Gamma_{12}^2(\Gamma_{12}^2+\Gamma_{11}^1)\\&\quad-2(\Gamma_{11}^1\Gamma_{12}^2+\Gamma_{11}^1\Gamma_{12}^1)].\end{aligned}\tag{6.1}$$

以下来计算 (2) 的右端.

$$\begin{aligned}
&\frac{1}{\sqrt{EG-F^2}}\left[\frac{\partial}{\partial v}\left(\frac{\sqrt{EG-F^2}}{E}\Gamma_{11}^2\right)-\frac{\partial}{\partial u}\left(\frac{\sqrt{EG-F^2}}{E}\Gamma_{12}^2\right)\right]\\
=&\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u^2}\left(\frac{\sqrt{g}}{g_{11}}\Gamma_{11}^2\right)-\frac{\partial}{\partial u^1}\left(\frac{\sqrt{g}}{g_{11}}\Gamma_{12}^2\right)\right]\\
=&\frac{1}{\sqrt{g}}\left[\frac{\frac{\partial g_{11}}{\partial u^2}\sqrt{g}-\frac{\partial\sqrt{g}}{\partial u^2}\cdot g_{11}}{(g_{11})^2}\Gamma_{11}^2+\frac{\sqrt{g}}{g_{11}}\frac{\partial\Gamma_{11}^2}{\partial u^2}-\right.\\
&\left.\qquad\frac{\frac{\partial g_{11}}{\partial u^1}\sqrt{g}-g_{11}\frac{\partial\sqrt{g}}{\partial u^1}}{(g_{11})^2}\Gamma_{11}^2-\frac{\sqrt{g}}{g_{11}}\cdot\frac{\partial\Gamma_{12}^2}{\partial u^1}\right]\\
=&\frac{1}{\sqrt{g}}\left[\frac{\sqrt{g}\frac{\partial g_{11}}{\partial u^2}}{(g_{11})^2}\Gamma_{11}^2-\frac{\sqrt{g}\frac{\partial g_{11}}{\partial u^1}}{(g_{11})^2}\Gamma_{12}^2\right]+\\
&\frac{1}{g_{11}}\left(\frac{\partial\sqrt{g}}{\partial u^1}\Gamma_{12}^2-\frac{\partial\sqrt{g}}{\partial u^2}\Gamma_{11}^2\right)+\frac{\sqrt{g}}{g_{11}}\left(\frac{\partial\Gamma_{11}^2}{\partial u^2}-\frac{\partial\Gamma_{12}^2}{\partial u^1}\right).
\end{aligned}\tag{6.2}$$

由于

$$\begin{aligned}
\frac{\partial\sqrt{g}}{\partial u^1}&=\frac{\partial}{\partial u^1}\sqrt{EG-F^2}=\frac{1}{2\sqrt{EG-F^2}}[E_uG+EG_u-2FF_u]\\
&=\frac{1}{2\sqrt{EG-F^2}}[GE_u-2FF_u+FF_v+EG_v-FF_v]\\
&=(\Gamma_{11}^1+\Gamma_{12}^2)\sqrt{g}.
\end{aligned}\tag{6.3}$$

同理

$$\begin{aligned}
\frac{\partial\sqrt{g}}{\partial u^2}&=\frac{\partial}{\partial u^2}\sqrt{EG-F^2}=\frac{1}{2\sqrt{EG-F^2}}[E_vG+EG_v-2FF_v]\\
&=\frac{1}{2\sqrt{EG-F^2}}[GE_v-2FF_v-FG_u+EG_v-FG_u]\\
&=(\Gamma_{12}^1+\Gamma_{22}^2)\sqrt{g}.
\end{aligned}\tag{6.4}$$

注意到

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial u^l}&=\boldsymbol{r}_{il}\cdot\mathbf{r}_j+r_i\cdot\mathbf{r}_{jk}=[il,j]+[jk,i]\\
&=\sum_m\Gamma_{il}^m g_{mj}+\sum_m\Gamma_{jl}^m g_{mi},
\end{aligned}$$

所以

$$\begin{aligned}\Gamma_{11}^2\frac{\partial g_{11}}{\partial u^2} &= \Gamma_{11}^2(\Gamma_{12}^1 g_{11}+\Gamma_{12}^2 g_{21}+\Gamma_{12}^1 g_{11}+\Gamma_{12}^2 g_{21}) \\ &= 2(\Gamma_{12}^1 g_{11}+g_{12}\Gamma_{12}^2)\Gamma_{11}^2.\end{aligned}$$

同理

$$\Gamma_{12}^2\frac{\partial g_{11}}{\partial u^1} = 2(\Gamma_{11}^1 g_{11}+g_{12}\Gamma_{12}^2)\Gamma_{12}^2.$$

因此

$$\Gamma_{11}^2\frac{\partial g_{11}}{\partial u^2}-\Gamma_{12}^2\frac{\partial g_{11}}{\partial u^1} = 2g_{11}(\Gamma_{11}^1\Gamma_{12}^2-\Gamma_{12}^1\Gamma_{11}^2). \tag{6.5}$$

将 (6.3)、(6.4)、(6.5) 式代入 (6.2) 式, 整理后得到与 (6.1) 式右端相同的表达式, 从而证明了 (2) 的结论.

(3) 与 (2) 的证明类似.

(4) 和 (5) 利用正交网的计算公式 $K=\frac{-1}{\sqrt{EG}}\left[\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u+\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v\right]$ 直接验证, 或利用本题 (1) 的结论直接验证即可.

7. 如果曲面的第一基本形式是 $\mathrm{d}s^2=\frac{\mathrm{d}u^2+\mathrm{d}v^2}{(u^2+v^2+C)^2}$, 计算第二类克氏记号.

解: 因为

$$E=G=\frac{1}{(u^2+v^2+C)^2},\quad F=0,$$

所以

$$E_u=G_u=\frac{-4u}{(u^2+v^2+C)^3},$$

$$E_v=G_v=\frac{-4v}{(u^2+v^2+C)^3},$$

从而

$$\begin{aligned}\Gamma_{11}^1&=\frac{E_u}{2E}=\frac{-2u}{u^2+v^2+C}, & \Gamma_{11}^2&=\frac{-E_v}{2G}=\frac{2v}{u^2+v^2+C},\\ \Gamma_{12}^1&=\frac{E_v}{2E}=\frac{-2v}{u^2+v^2+C}, & \Gamma_{12}^2&=\frac{G_u}{2G}=\frac{-2u}{u^2+v^2+C},\\ \Gamma_{22}^1&=\frac{-G_u}{2E}=\frac{2u}{u^2+v^2+C}, & \Gamma_{22}^2&=\frac{G_v}{2G}=\frac{-2v}{u^2+v^2+C}.\end{aligned}$$

8. 求证第一基本形式是 $\mathrm{d}s^2=\frac{\mathrm{d}u^2+\mathrm{d}v^2}{(u^2+v^2+C)^2}$ 的曲面有常高斯曲率.

证明：因 $F=0$, 将 $E=G=\frac{1}{(u^2+v^2+C)^2}$ 代入正交网时Gauss曲率的计算公式 $K=\frac{-1}{\sqrt{EG}}\left[\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u+\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v\right]$, 直接计算可得曲面的Gauss曲率 $K=4C$.

9. 求以 $E=1,\ F=0,\ G=1,\ L=-1,\ M=0,\ N=0$ 为第一、第二类基本量的曲面.

解：先求第二类克氏符号:

$$\Gamma_{11}^1=\frac{E_u}{2E}=0,\quad \Gamma_{11}^2=\frac{-E_v}{2G}=0,$$

$$\Gamma_{12}^1=\frac{E_v}{2E}=0,\quad \Gamma_{12}^2=\frac{G_u}{2G}=0,$$

$$\Gamma_{22}^1=\frac{-G_u}{2E}=0,\quad \Gamma_{22}^2=\frac{G_v}{2G}=0.$$

再计算 μ_i^j:

$$\mu_1^1=-\frac{L}{E}=1,\qquad \mu_1^2=-\frac{M}{G}=0,$$

$$\mu_2^1=-\frac{M}{E}=0,\qquad \mu_2^2=-\frac{N}{G}=0.$$

可以验证它们满足高斯-科达齐方程, 于是所求平面存在. 同时曲面的运动方程如下:

$$\boldsymbol{r}_{uu}=-n, \tag{9.1}$$

$$\boldsymbol{r}_{uv}=0, \tag{9.2}$$

$$\boldsymbol{r}_{vv}=0, \tag{9.3}$$

$$\boldsymbol{n}_u=r_u, \tag{9.4}$$

$$\boldsymbol{n}_v=0. \tag{9.5}$$

由方程(9.1)与(9.4)得

$$\boldsymbol{r}_{uuu}+\boldsymbol{r}_u=0.$$

积分得

$$\boldsymbol{r}=e_1(v)\sin u+e_2(v)\cos u+e_3(v).$$

于是

$$\boldsymbol{r}_u = e_1(v)\cos u - e_2(v)\sin u,$$

$$\boldsymbol{r}_{uv} = e_1'(v)\cos u - e_2'(v)\sin u.$$

根据上式与方程(9.2)知

$$e_1'(v)\cos u - e_2'(v)\sin u = 0,$$

即

$$e_1'(v) = e_2'(v)\tan u,$$

由于$e_1(v)$, $e_2(v)$只与v有关, 故上式成立当且仅当$e_1'(v) = e_2'(v) = 0$, 所以$e_1(v)$, $e_2(v)$是常向量.

$$\boldsymbol{r} = e_1\sin u + e_2\cos u + e_3(v).$$

由此得

$$\boldsymbol{r}_v = e_3'(v), \quad r_{vv} = e_3''(v),$$

根据方程(9.3),

$$\boldsymbol{r}_{vv} = e_3''(v) = 0,$$

故有$e_3(v) = a + bv$(其中a, b是常向量).

所求曲面的方程为

$$\boldsymbol{r} = e_1\sin u + e_2\cos u + (a + bv).$$

10. 证明不存在曲面, 使$E = G = 1$, $F = 0$, $L = 1$, $M = 0$, $N = -1$.

证明: 假设题设曲面存在, 则可以计算得到:

$$\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = \Gamma_{22}^2 = 0,$$

于是, $R_{mijk} = 0$, 特别$R_{1212} = 0$. 但是由高斯方程

$$R_{1212} = L_{21}L_{12} - L_{22}L_{11} = M^2 - LN = 1 \neq 0,$$

因此, 满足题设条件的曲面不存在.