曲线论综合练习题

1. Show that the curve $C: r(t) = \{t + \sqrt{3}\sin t, 2\cos t, \sqrt{3}t - \sin t\}$ and the curve $\tilde{C}: \tilde{r}(t) = \{2\cos\frac{t}{2}, 2\sin\frac{t}{2}, -t\}$ are congruent.

2. Determine a curve $r(s)$ with $k(s) > 0$, $\tau(s) > 0$ and $\tau(s) = c \cdot k(s)$ for some nonzero constant c, $-\infty < s < \infty$.

3. Prove that a necessary and sufficient condition for a curve $r(s)$ with curvature $k(s) \neq 0$ and torsion $\tau(s) \neq 0$ be a spherical curve is

$$\frac{\tau(s)}{k(s)} + \frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{1}{\tau(s)} \cdot \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{k(s)}\right]\right\} = 0,$$

where s is the arc-length parameter of the curve $r(s)$.

4. Let $\alpha(s)$, $\beta(s)$ and $\gamma(s)$ be the tangent vector, principal normal vector and binormal vector of the curve C respectively, then the following curves

$$C_1: r_1(s) = \alpha(s), \quad C_2: r_2(s) = \beta(s), \quad C_3: r_3(s) = \gamma(s)$$

be called the tangent spherical image, principal normal spherical image and binormal spherical image respectively. Show that

(1) If $s_i (1 \leq i \leq 3)$ be the arc-length parameters of the curves C_i, then

$$\left|\frac{\mathrm{d}s_1}{\mathrm{d}s}\right| = k, \quad \left|\frac{\mathrm{d}s_2}{\mathrm{d}s}\right| = \sqrt{k^2 + \tau^2}, \quad \left|\frac{\mathrm{d}s_3}{\mathrm{d}s}\right| = |\tau|,$$

where s be the arc-length of the curve C, $k(s)$ and $\tau(s)$ be the curvature and the torsion of the curve C respectively;

(2) The necessary and sufficient conditions for the tangent spherical image to be constant curve are the curve C be straight line; C be plane curve is the necessary and sufficient condition for the spherical indicatric of the tangent to be a great circle or parts of it;

(3) The spherical indicatric of the principal normal always can't be constant curve;

(4) C be a plane curve is the necessary and sufficient condition for the spherical indicatric of the binormal to be constant curve;

(5) Show that the curvature and torsion of the tangent spherical indicatric are $k_1 = \sqrt{1 + (\tau/k)^2}$ and $\tau_1 = \frac{\frac{\mathrm{d}}{\mathrm{d}s}(\tau/k)}{k(1+(\tau/k)^2)}$ respectively.

5. (1) All the tangents to a curve go through one fixed point if and only if the curve is straight line;

(2) All the osculating planes pass through one fixed point if and only if the curve is plane curve;

(3) All the principal normal lines have a common point of intersection if and only if the curve is a circle;

(4) All the normal planes to a curve go through a given fixed point if and only if the image of the curve lies on a sphere.

6. For the regular curve $C: \ r = r(s)$ parameterized by arc-length s, suppose that its curvature $k(s) > 0$ and torsion $\tau(s) > 0$. Denoted by $\gamma(s)$ be the binormal vector of the curve $r(s)$. We define new curve $\tilde{C}$:

$$\tilde{r}(s) = \int_0^s \gamma(u)\mathrm{d}u$$

(1) Show that s is also the arc length of $\tilde{C}$, and $\tilde{k} = \tau, \ \tilde{\tau} = k$;

(2) Find the Frenet frame on the curve $\tilde{C}$ at any point.

7. Determine the curvature and torsion of the spherical curve

$$\begin{cases} x^2 + y^2 + z^2 = 9 \\ x^2 - y^2 = 3 \end{cases}$$

at the point $P_0(2, 1, 2)$.

8. (1) The locus of the center of curvature of the circular helix be also a circular helix;

(2) The locus of the center of curvature of the curve with constant curvature and non-zero torsion is also a curve with constant curvature.

9. (1) Determine the curvature of the curve $r(-t)$ as a plane curve;

(2) Determine the curvature and torsion of the curve $r(-t)$ as a space curve;

(3) Determine the curvature and torsion of the symmetric curve of $r(t)$ about the origin;