# Chapter 1

# Local Curve Theory

We shall begin our study of differential geometry with an investigation of curves in three-dimensional Euclidean space, $\mathbf{R}^3$. This is a good place to start for four reasons: (1) since curves in $\mathbf{R}^3$ are easy to draw and visualize, we can develop some geometric insight into the subject by looking at examples; (2) it is a very complete subject—the Frenet-Serret apparatus of a curve completely determines the local geometry of curves and gives a complete set of invariants for the problem of determining whether two curves are the same; (3) the theory of curves will introduce us to some techniques that are the mainstay of modern differential geometry (e.g., linear algebra); (4) we shall base our study of surfaces in Chapter 2 on the behavior of curves on the surfaces.

The history of the theory of curves (and of all differential geometry) is a fascinating one. Suffice it to say at this point that the many results in the theory of curves in $\mathbf{R}^3$, which we discuss in this chapter, were initiated by G.Monge (1746–1896) and his school (Meusnier, Lancret, and Dupin). Our approach is duo to G.Darboux (1842–1917) whose idea of moving frames unifies a great deal of the classical theory of curves. He accomplished this in 1887–1896. It is interesting to note that the approach to the theory of surfaces what we will take in Chapter 2 is that of K.Gauss in 1827. This anomaly of dates is not due to the fact that the curve theory is more difficult than the theory of surface (just the opposite is true) but rather to the pervasive genius of Gauss.

There are two ways to think of curves. The first is as a geometric set of points, or locus. When this is the case, we refer to a *geometric curve*, or the *geometric shape* of the curve. Intuitively we are thinking of a curve as the path traced out by a particle moving in $\mathbf{R}^3$. The second way of thinking of a curve is as a function of some parameter, say $t$. Intuitively it is not always enough to know where a particle went—we also want to know when it go there. (The parameter $t$ is often thought of as time.) It is necessary to view curves the second way if we are to apply the techniques of calculus to describe the geometric behavior of a curve. This means that we must pay careful attention to how the curve is parameterized(e.g., if you change the parameter you also change the velocity vector field to the curve). However, we are also interested in geometric properties of curves(e.g., arc length, tangent vector field). These should not depend on the way a curve is parameterized as a function but only on the the image set of the function, that is, only on the geometric shape. Thus we shall ask whether our constructions and descriptions depend upon the parameterizations.

In Section 1–1 we define and give examples of parameterized curves. In Section 1–2 we introduce a particularly useful parametrization— that by arc length. Section 1–3 develops the Frenet-Serret apparatus which is the basic tool in the study of curves. It consists of three vector fields along the given curve(the tangent $\mathbf{T}$, the normal $\mathbf{N}$, and the binormal $\mathbf{B}$) and two scalar-valued functions(the curvature $\kappa$ and the the torsion $\tau$). The Frenet-Serret Theorem is proved in the fourth section. This theorem express the derivatives of $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$ in terms of $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$. We then make several applications. Section 1–5 gives the Fundamental Theorem of Curves and shows that the Frenet-Serret apparatus does completely determine the geometry of the curve. Finally, in Section 1–6 we develop the necessary techniques for computing the Frenet-Serret apparatus for curves which are not parameterized by arc length.

## 1–1. BASIC DEFINITIONS AND EXAMPLES

Our study of curves will be restricted to a certain class of curves in $\mathbf{R}^3$. Not only do we want a curve to be described by a differential function so that we may use calculus to describe the geometry, we also want to avoid certain pathologies and technicalities. If $d\boldsymbol{\alpha}/dt = 0$ on an interval, then $\boldsymbol{\alpha}(t)$ is constant in that interval, which is geometrically very uninteresting. If $d\boldsymbol{\alpha}/dt$ is zero at some point, then the graph of $\boldsymbol{\alpha}$ can have a sharp corner, which is geometrically unappealing. (Consider the graph of $\boldsymbol{\alpha}(t) = (t^2, t^3, 0)$.) Because of these considerations we will only work with regular curves.

Definition 1 A *regular curve* in $\mathbf{R}^3$ is a function $\boldsymbol{\alpha} : (a, b) \to \mathbf{R}^3$ which is a class $C^k$ for some $k \geq 1$ and for which $d\boldsymbol{\alpha}/dt \neq 0$ for all $t \in (a, b)$.

In this text, a regular curve will be assumed to be of class $C^3$ unless stated otherwise.

Note that from this point of view the curve is the function and *not* the image set (geometric curve). Two different curves may have the same image set (see Examples 1.2 and 1.4 below). A regular curve need not be one-to-one.

Given a regular curve $\boldsymbol{\alpha}(t)$, we can define some vector fields along $\boldsymbol{\alpha}$. This means that for each $t$ we will have a 3-vector $\mathbf{v}(t)$. The reader should think of the tail of $\mathbf{v}(t)$ to be at the point $\boldsymbol{\alpha}(t)$. The mapping $t \to \mathbf{v}(t)$ is a vector-valued function and so we may use the material of vector calculus.

DEFINITION 2 The *velocity vector* of a regular curve $\boldsymbol{\alpha}(t)$ at $t = t_0$ is the derivative $d\boldsymbol{\alpha}/dt$ evaluated at $t = t_0$. The *velocity vector field* is the vector-valued function $d\boldsymbol{\alpha}/dt$. The *speed* of $\boldsymbol{\alpha}(t)$ at $t = t_0$ is the length of he velocity vector at $t = t_0$, $|(d\boldsymbol{\alpha}/dt)(t_0)|$.

If we view the curve as the path of a moving particle, the velocity vector at $t = t_0$ points in the direction that the particle is moving at time $t = t_0$. The regularity condition says that the speed is always nonzero—the particle never stops moving, even instantaneously.

DEFINITION 3 The *tangent vector field* to a regular curve $\boldsymbol{\alpha}(t)$ is the vector-valued function $\mathbf{T}(t) = \frac{d\boldsymbol{\alpha}/dt}{|d\boldsymbol{\alpha}/dt|}$.

Note that we are able to define $\mathbf{T}$ (i.e., divide by $|d\boldsymbol{\alpha}/dt|$) precisely because of the regularity condition. $\mathbf{T}$ is the unit vector in the direction of the velocity vector.

We shall see later in this section that $\mathbf{T}$ is a geometric quantity: it depends only on the image set of $\boldsymbol{\alpha}$ and not the particular way this set is parameterized.

For each value of $t$, say $t_0$, there is a unique straight line through $\boldsymbol{\alpha}(t_0)$ parallel to $\mathbf{T}(t_0)$. This line is a linear approximation of the curve near $\boldsymbol{\alpha}(t_0)$. This is an example of one of the basic techniques in differential geometry: an object of study (a curve) is replaced by a linear approximation (a tangent line). This is done because linear mathematics is so much better understood than nonlinear mathematics. More formally:

DEFINITION 4 The *tangent line* to a regular curve $\boldsymbol{\alpha}$ at the point $t = t_0$ is the straight line

$$l = \{\mathbf{w} \in \mathbf{R}^3 \mid \mathbf{w} = \boldsymbol{\alpha}(t_0) + \lambda\mathbf{T}(t_0), \lambda \in \mathbf{R}\}.$$

Note that the tangent line is a subset of $\mathbf{R}^3$ which contains the point $\boldsymbol{\alpha}(t_0)$ and actually is a straight line. Intuitively it is the line that most nearly approximation the curve near $\boldsymbol{\alpha}(t_0)$.

Since $d\boldsymbol{\alpha}/dt \neq 0$ and $d\boldsymbol{\alpha}/dt = |d\boldsymbol{\alpha}/dt|\mathbf{T}$, the tangent line at $t = t_0$ is also given by

$$\{\mathbf{w} \in \mathbf{R}^3 \mid \mathbf{w} = \boldsymbol{\alpha}(t_0) + \mu\frac{d\boldsymbol{\alpha}}{dt}(t_0), \mu \in \mathbf{R}\}.$$

EXAMPLE 1.1 Let $\mathbf{u}$ and $\mathbf{v}$ be fixed vectors in $\mathbf{R}^3$. Then the curve $\boldsymbol{\alpha} : \mathbf{R} \to \mathbf{R}^3$ given by $\boldsymbol{\alpha}(t) = \mathbf{u} + t\mathbf{v}$ is a regular curve if and only if $\mathbf{v} \neq 0$. In this case it is a straight line and $d\boldsymbol{\alpha}/dt = \mathbf{v}$. The tangent line at each point is the given straight line and $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$.

EXAMPLE 1.2 Let $\boldsymbol{\alpha} : \mathbf{R} \to \mathbf{R}^3$ be given by $\boldsymbol{\alpha}(t) = (t, 0, 0)$. This is a special case of Example 1.1.

EXAMPLE 1.3 Let $\boldsymbol{\alpha} : \mathbf{R} \to \mathbf{R}^3$ be given by $\boldsymbol{\alpha}(t) = (t^3, 0, 0)$. $d\boldsymbol{\alpha}/dt = (3t^2, 0, 0)$, which is zero at $t = 0$. $\boldsymbol{\alpha}$ is not a regular curve, even though its image is the same as the curve in Example 1.2!

EXAMPLE 1.4 Let $\boldsymbol{\alpha} : \mathbf{R} \to \mathbf{R}^3$ be given by $\boldsymbol{\alpha}(t) = (t^3 + t, 0, 0)$. $\frac{d\boldsymbol{\alpha}}{dt} = (3t^2 + 1, 0, 0) \neq 0$. This is a regular curve whose image set is the same as the curve in Example 1.2.

EXAMPLE 1.5 Let $g : \mathbf{R} \to \mathbf{R}$ be a differential function. Let $\boldsymbol{\alpha}(t) = (t, g(t), 0)$. Then $d\boldsymbol{\alpha}/dt = (1, g'(t), 0) \neq 0$ and $\boldsymbol{\alpha}$ is a regular curve. $\boldsymbol{\alpha}$ is the graph of $g$ except for the extra (third) coordinate. The tangent line at $t = t_0$ is $\{(t_0 + \lambda, g(t_0) + \lambda g'(t_0), 0) \mid \lambda \in \mathbf{R}\}$. In terms of $t, y, z$ coordinates this line is $z = 0, y - g(t_0) = (t - t_0)g'(t_0)$, which should be familiar from Calculus I as the equation of the tangent line to the graph of $g(t)$ at $t = t_0$. (Remember $t = x$ in this example.)

EXAMPLE 1.6 Let $\boldsymbol{\alpha} : \mathbf{R} \to \mathbf{R}^3$ be given by $\boldsymbol{\alpha}(t) = (r\cos t, r\sin t, ht)$, where $h > 0$ and $r > 0$. This is called a *right circular helix.* (If $h < 0$ it would be a *left circular helix.*) Circular refer to the fact that the projection in the $(x, y)$ plane is a circle. Since $d\boldsymbol{\alpha}/dt = (-r\sin t, r\cos t, h) \neq 0$, $\boldsymbol{\alpha}$ is a regular curve. At $t = t_0, \mathbf{T} = (-r\sin t_0, r\cos t_0, h)/\sqrt{r^2 + h^2}$.

In Examples 1.2 and 1.3 we saw a situation where the same image set (the $x$-axis) was given two different parameterizations, one of which was not regular. We wish to know what parameterizations can be used to describe a given image curve.

DEFINITION 5 A *reparameterization* of a curve $\boldsymbol{\alpha} : (a, b) \to \mathbf{R}^3$ is a one-to-one onto function $g : (c, d) \to (a, b)$ such that both $g$ and its inverse $h : (a, b) \to (c, d)$ are of class $C^k$ for some $k \geq 1$.

What we have in mind is the new curve $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ g$. If $r$ denotes the variable in the interval $(c, d)$, then $d\boldsymbol{\beta}/dr = (d\boldsymbol{\alpha}/dt)(dg/dr)$ by the chain rule. $\boldsymbol{\beta}$ is thus regular if $\boldsymbol{\alpha}$ is regular and $dg/dr \neq 0$. But $g(h(t)) = t$ so that by the chain rule curve $(dg/dr)(dh/dt) = 1$ and $dg/dr \neq 0$. Thus the composition of a regular curve with a reparameterization yields a regular curve. Note that if $\boldsymbol{\alpha}$ is of class $C^m$ and $g$ is of class $C^k$, then $\boldsymbol{\beta}$ is of class $C^n$ with $n = \min(m, k)$.

The image of a curve and any of its reparametreization are the same. This means that any quality which stays the same when we change the parameters (i.e., make a reparameterization) is a quality which depends only on the geometric shape of the curve. Briefly, the quality is a geometric invariant. We will show below that the tangent line to a regular curve is such a geometric invariant after we give some examples.

EXAMPLE 1.7 Let $g : (0,1) \to (1,2)$ be given by $g(r) = 1 + r^2$. $g$ is one-to-one and onto with inverse $h(t) = \sqrt{t-1}$. $g$ is infinitely differentiable on $(0,1)$ and so is $h$ on $(1,2)$. Thus $g$ is a reparameterization of any regular curve on $(1,2)$.

EXAMPLE 1.8 Let $g : \mathbf{R} \to \mathbf{R}$ be given by $g(r) = r^3$. $g$ is one-to-one, onto, and infinitely differentiable. However, $h(t) = t^{1/3}$ is the inverse and $h'(0)$ does not exist, so that $h$ is not $C^1$. This is one reason why Example 1.3 was not a regular curve.

Now we shall show that the tangent vector field is a geometric property of the image set of a regular curve and does not depend on the parametrization. This means that the tangent line to a curve is a geometric property also.

PROPOSITION 1.1 Let $\boldsymbol{\alpha} : (a,b) \to \mathbf{R}^3$ be a regular curve and let $g : (c,d) \to (a,b)$ be a reparameterization. Set $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ g$. If $t_0 = g(r_0)$, the tangent vector field $\mathbf{T}$ of $\boldsymbol{\alpha}$ at $t = t_0$ and the tangent vector field $\mathbf{T}^*$ of $\boldsymbol{\beta}$ at $r_0$ satisfy $\mathbf{T}^* = \pm\mathbf{T}$.

*Proof:*

$$\mathbf{T}^* = \frac{\frac{d\boldsymbol{\beta}}{dr}}{\left|\frac{d\boldsymbol{\beta}}{dr}\right|} = \frac{\frac{d\boldsymbol{\alpha}}{dt}\frac{dg}{dr}}{\left|\frac{d\boldsymbol{\alpha}}{dt}\right|\left|\frac{dg}{dr}\right|} = \frac{\frac{d\boldsymbol{\alpha}}{dt}}{\left|\frac{d\boldsymbol{\alpha}}{dt}\right|}\frac{\frac{dg}{dr}}{\left|\frac{dg}{dr}\right|} = (\mathbf{T})(\pm 1) = \mathbf{T}.$$

□

Note that $\mathbf{T}^* = \mathbf{T}$ if $dg/dr > 0$ ($g$ is increasing) and $\mathbf{T}^* = -\mathbf{T}$ if $dg/dr < 0$ ($g$ is decreasing). Geometrically the difference is whether $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ indicate particles moving along the image curve in the same or opposite directions.

## 1–2. ARC LENGTH

Sometimes it is useful for technical reasons to consider curves with end points, that is, curves defined on closed intervals:

DEFINITION 1 A *regular curve segment* is a function $\boldsymbol{\alpha} : [a,b] \to \mathbf{R}^3$ together with an open interval $(c,d)$ with $c < a < b < d$, and a regular curve $\boldsymbol{\gamma} : (c,d) \to \mathbf{R}^3$ such that $\boldsymbol{\alpha}(t) = \boldsymbol{\gamma}(t)$ for all $t \in [a,b]$.

Thus a curve segment is a curve defined on a closed interval which can be extended to a curve on a slightly larger open interval. In this case it is possible to talk about $d\boldsymbol{\alpha}/dt$ at the end points of the curve segment because we define $(d\boldsymbol{\alpha}/dt)(a)$ to be $(d\boldsymbol{\gamma}/dt)(a)$ and $(d\boldsymbol{\alpha}/dt)(b)$ to be $(d\boldsymbol{\gamma}/dt)(b)$.

Now we shall define the length of a curve segment. The intuitive justification for what we do is as follows. If $\boldsymbol{\alpha}(t)$ is viewed as the path of a particle moving in space, then $|d\boldsymbol{\alpha}/dt|$ is the speed of the particle as a function of time. The integral of speed should be the distance traveled by the particle just as it is in one dimension.

DEFINITION 2 The *length* of a regular curve segment $\boldsymbol{\alpha} : [a, b] \to \mathbf{R}^3$ is

$$\int_a^b \left| \frac{d\boldsymbol{\alpha}}{dt} \right| dt.$$

Note that this is really the familiar formula for the length of a curve in $\mathbf{R}^3$: if $\boldsymbol{\alpha}(t) = (x(t), y(t), z(t))$, then $|d\boldsymbol{\alpha}/dt| = \sqrt{(x')^2 + (y')^2 + (z')^2}$, so that the length of the curve is given by $\int_a^b \sqrt{(x')^2 + (y')^2 + (z')^2}\, dt$.

It makes sense to talk about reparameterization of a curve segment. One should hope that the length of a curve is a geometric property and does not depend on the choice of parametrization. This is the content of the next proposition.

PROPOSITION 2.1 Let $g : [c, d] \to [a, b]$ be a reparameterization of a curve segment $\boldsymbol{\alpha} : [a, b] \to \mathbf{R}^3$. Then the length of $\boldsymbol{\alpha}$ is equal to the length of $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ g$.

*Proof:* The length of $\boldsymbol{\beta}$ is

$$\int_c^d \left| \frac{d\boldsymbol{\beta}}{dr} \right| dr = \int_c^d \left| \left( \frac{d\boldsymbol{\alpha}}{dt} \right) \left( \frac{dg}{dr} \right) \right| dr = \int_c^d \left| \frac{d\boldsymbol{\alpha}}{dt} \right| \left| \frac{dg}{dr} \right| dr.$$

*Case 1:* If $dg/dr > 0$, then $|dg/dr| = dg/dr$, $g(c) = a, g(d) = b$, and $\int_c^d |d\boldsymbol{\alpha}/dt||dg/dr|dr = \int_c^d |d\boldsymbol{\alpha}/dt|(dg/dr)dr = \int_a^b |d\boldsymbol{\alpha}/dt|dt$ by the substitution rule of integral calculus.

*Case 2:* If $dg/dr < 0$, the proof is similar, using $|dg/dr| = -dg/dr$, $g(c) = b$, and $g(d) = a$.

In both cases we have that the length of $\boldsymbol{\alpha}$ equals the length of $\boldsymbol{\beta}$. □

Note that the definition of the length of a curve does not really require $\boldsymbol{\alpha}$ to regular to make sense—it is sufficient for $\boldsymbol{\alpha}$ to be of class $C^1$. However, if $\boldsymbol{\alpha}$ is not regular, some segments of the curve may be traversed twice and the formula will count the doubled section twice.

Using the concept of length of a curve, we are able to define an important way to reparameterize a curve. Let $\boldsymbol{\alpha} : (a, b) \to \mathbf{R}^3$ be a regular curve and let

$t_0 \in (a, b)$. Set $h(t) = \int_{t_0}^{t} |d\boldsymbol{\alpha}/dt|dt$. $s = h(t)$ is called *arc length along* $\boldsymbol{\alpha}$. It actually measures signed arc length along $\boldsymbol{\alpha}$ from $\boldsymbol{\alpha}(t_0)$ with $h(t) < 0$ if $t < t_0$ and $h(t) > 0$ if $t > t_0$.

THEOREM 2.2 *$h$ is a one-to-one function mapping $(a, b)$ onto some interval $(c, d)$ and is a reparameterization.*

*Proof:* By the fundamental theorem of calculus, $dh/dt = |d\boldsymbol{\alpha}/dt| > 0$ (since $\boldsymbol{\alpha}$ is regular). Thus $h$ is increasing and so one-to-one. It is easy to check that if $\boldsymbol{\alpha}$ is of class $C^k$ so is $h$. Let $g : (c, d) \to (a, b)$ be the inverse of $h$ and denote the parameter in $(c, d)$ by $s$. This means of course that $g(s) = t$ if and only if $h(t) = s$ so that $s$ is the arc length parameter. Because $g$ and $h$ are inverse functions $dg/ds = 1/(dh/dt)$, where the right-hand side is evaluated at $t = g(s)$. This quotient makes sense since $dh/dt \neq 0$. $g$ can be differentiated as often as $h$ can. Thus $h$ is a reparameterization. □

As we pointed out in the proof of the above theorem, $s$ is the arc length. By using $g(s)$, any regular curve $\boldsymbol{\alpha}$ can be reparameterized in terms of arc length from a point. Once this has been done we say that the curve has been *parameterized by arc length.* The importance of a curve being parameterized by arc length is carried in the observation that its velocity vector field is its tangent vector field, as may be seen in the following computation.

If $\boldsymbol{\beta}(s)$ is parameterized by arc length, then $s = \int_0^s |d\boldsymbol{\beta}/d\sigma|d\sigma$. By the fundamental theorem of calculus, $1 = \frac{d}{ds}(\int_0^s |d\boldsymbol{\beta}/d\sigma|d\sigma) = |d\boldsymbol{\beta}/d\sigma|$ at $\sigma = s$, that is $1 = |d\boldsymbol{\beta}/ds|$. Hence the velocity vector field $d\boldsymbol{\beta}/ds$ is a unit vector field and is thus $\mathbf{T}$. When a curve $\boldsymbol{\beta}$ is parameterized by arc length (or equivalently if its velocity vector field is $\mathbf{T}$) we say that $\boldsymbol{\beta}$ is a *unit speed curve.*

The preceding paragraph and the proof of Theorem 2.2 contain some important facts which we now isolate. We shall assume that 0 is in the domain of $\boldsymbol{\alpha}(t)$ and base arc length at 0.

COROLLARY 2.3 *If $\boldsymbol{\alpha}(t)$ is a regular curve with arc length $s = s(t)$, then*

(a) $s = s(t) = \int_0^t |d\boldsymbol{\alpha}/dt|dt$;
(b) $ds/dt = |d\boldsymbol{\alpha}/dt|$;
(c) $d\boldsymbol{\alpha}/dt = (ds/dt)\mathbf{T}$; and
(d) $\mathbf{T} = d\boldsymbol{\alpha}/ds$.

To take a given regular curve $\boldsymbol{\alpha}$ and reparameterized it by arc length, while always possible in theory, may be very difficult in practice. There are two obstacles to such a program. In the first place, the integral

$$h(t) = \int_{t_0}^{t} \left| \frac{d\boldsymbol{\alpha}}{dt} \right| dt$$

may not be elementary (see Example 2.3) and hence not computable. Second, even if $h(t)$ can be determined, it may not possible to find the inverse function $g(s)$ (see Example 2.4).

EXAMPLE 2.1 Let $\boldsymbol{\alpha}(t) = \mathbf{u} + t\mathbf{v}$ be the straight line of Example 1.1. $d\boldsymbol{\alpha}/dt = \mathbf{v}$, $s = h(t) = \int_0^t |\mathbf{v}| dt = t\mathbf{v}$. Thus $t = g(s) = s/\mathbf{v}$. $\boldsymbol{\beta}(s) = u + s\mathbf{v}/|\mathbf{v}|$ gives the unit speed parametrization of a straight line. Note that the tangent vector field to $\boldsymbol{\alpha}$ is $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$, and $d\mathbf{T}/ds = 0$.

EXAMPLE 2.2 Let $\boldsymbol{\alpha}(t) = (r\cos t, r\sin t, 0)$ with $r > 0$.

$$\frac{d\boldsymbol{\alpha}}{dt} = (-r\sin t, r\cos t, 0)$$

and $|d\boldsymbol{\alpha}/dt| = r$. $s = h(t) = rt$ and $t = g(s) = s/r$.

$$\boldsymbol{\beta}(s) = (r\cos(s/r), r\sin(s/r), 0)$$

is the unit speed parametrization of a circle of radius $r$. Note that the tangent vector field of $\boldsymbol{\alpha}$ is

$$\mathbf{T}(s) = (-\sin(s/r), \cos(s/r), 0)$$

and that

$$\frac{d\mathbf{T}}{ds} = \Big(-\frac{1}{r}\cos\big(\frac{s}{r}\big), -\frac{1}{r}\sin\big(\frac{s}{r}\big), 0\Big)$$

has length $1/r$.

EXAMPLE 2.3 If $\boldsymbol{\alpha}(t) = (2\sin t, \cos t, 0)$ is the ellipse, then

$$\frac{d\boldsymbol{\alpha}}{dt} = (2\cos t, -\sin t, 0).$$

$$\begin{aligned} \left|\frac{d\boldsymbol{\alpha}}{dt}\right| &= \sqrt{4\cos^2 t + \sin^2 t} \\ &= \sqrt{4 - 3\sin^2 t} = 2\sqrt{1 - \left(\tfrac{3}{4}\right)\sin^2 t}. \end{aligned}$$

But $\sqrt{1 - \left(\tfrac{3}{4}\right)\sin^2 t}$ does not have an elementary antiderivative and so the integration $h(t) = \int_0^t |d\boldsymbol{\alpha}/dt| dt$ cannot be carried out by using the fundamental theorem of calculus. (Definite integrals of this kind are called *elliptic integrals* because they can be interpreted as the arc length of an ellipse. Their values are tabulated in many books of mathematical tables.)

EXAMPLE 2.4 Let $\boldsymbol{\alpha}(t) = (t, t^2/2, 0)$. Then $d\boldsymbol{\alpha}/dt = (1, t, 0)$ and

$$\left|\frac{d\boldsymbol{\alpha}}{dt}\right| = \sqrt{1 + t^2}.$$

Hence

$$s = h(t) = \int_0^t \sqrt{1+\sigma^2}d\sigma = \tfrac{1}{2}(t\sqrt{1+t^2} + \ln(t+\sqrt{1+t^2})).$$

However, it is extremely difficult to find $t = g(s)$ from this equation. Not that $\boldsymbol{\alpha}$ is a parabola, a very simple curve geometrically!

What have we accomplished? Suppose that we want to study regular curves and we are interested only in their geometric shape (that is, we don't care about parametrization). Theorem 2.2 says that we may as well assume that the curve is parameterized by arc length. This will be a very useful technical device in setting up the Frenet-Serret apparatus in the next section.

## 1–3. CURVATURE AND THE FRENET-SERRET APPARATUS

This section is devoted almost entirely to making definitions and developing an intuitive feeling for these definitions. The reader should look again at the last paragraph of the previous section before starting on this section.

DEFINITION 1 A curve $\boldsymbol{\alpha} : (a, b) \to \mathbf{R}^3$ is a *unit speed curve* if $\left|\frac{d\boldsymbol{\alpha}}{dt}\right| = 1$.

Note that for a unit speed curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(t)$, the arc length $s = t - t_0$. We shall assume that $t_0$ has been chosen to be 0 (so that $s = t$) and will write $\boldsymbol{\alpha}$ as a function of $s$. Because of this convention we can write unambiguously $\boldsymbol{\alpha}' = \boldsymbol{\alpha}'(s) = \mathbf{T}(s)$ where $\mathbf{T}$ is, as always, the tangent vector field (see Corollary 2.3). For the rest of this section we shall assume that $\boldsymbol{\alpha}(s)$ is a unit speed curve. This assumption amounts to the philosophical statement that we are only interested in the geometric shape of a regular curve since any regular curve can be reparameterized by arc length(Theorem 2.2) and reparameterizing does not change the shape of a curve. In Section 1-6 we will compute the qualities defined below if the curve is not given in a unit speed parametrization.

We now motivate the definition of "curvature" (of a curve). "Curvature" will measure bending and will serve as central concept of study in this Lecture Notes (and, indeed, in all of differential geometry). The reader probably has some intuitive idea of what "curvature" is. Whether the definition of "curvature" is, it should satisfy two criteria: (1) the curvature of a straight line (Example 1.4) is zero; and (2) the curvature of a circle (Example 1.5) is the same at each point. In terms of the curvature measuring bending, (1) says that the straight line does not bed at all and (2) says that a circle has constant bending. What is it about a straight line that does not change (i.e., what might we choose to be measure of bending?)? A glance at Example 1.4 ($\boldsymbol{\alpha}(t) = mathbfu + t\mathbf{v}$) shows that the *tangent vector field of straight line does not change with the arc length s*. It is $\mathbf{v}/|\mathbf{v}|$, which is independent of $s$. A glance at Example 1.5 shows that the *tangent vector field of a circle of radius $r$ does change with $s$ but that its derivative has constant length* $1/r$. Because of thee consideration we are led to make the following definition.

DEFINITION 2 The *curvature* of a unit speed curve $\boldsymbol{\alpha}(s)$ is $\kappa(s) = |\mathbf{T}'(s)|$.

From above discussion, it is clear that $\kappa(s) = 0$ (for all $s$) if $\boldsymbol{\alpha}$ is a straight line and $\kappa(s) = 1/r$ (for all $s$) if $\boldsymbol{\alpha}$ is a circle of radius $r$. This last equality is especially appealing because it says that the smaller the radius is the larger the curvature is (that is, the faster or tighter the circle is bending), which conforms with our intuition.

We may also give a heuristic description of curvature for curves lying in the plane. If $\boldsymbol{\alpha}$ lies in the $(x, y)$ plane, then $\boldsymbol{\alpha}(s)$ takes the form

$$\boldsymbol{\alpha}(s) = (x(s), y(s), 0),$$

hence $\boldsymbol{\alpha}'(s) = \mathbf{T}(s) = (x'(s), y'(s), 0)$. We now let $\theta$ be the angle between the horizontal and the tangent vector field to $\boldsymbol{\alpha}$ at $s$.(*Technical remark:* This angle $\theta$ is not really well defined. It is defined only up to a multiple of $2\pi$. This is the reason we do not separate out what follows as a theorem but merely call it a heuristic description.)

Because of the description of the angle $\theta$ we have

$$x'(s) = \langle \mathbf{T}(s), (1, 0, 0) \rangle = \cos\theta(s)$$

and $\mathbf{T}(s) = (\cos\theta(s), \sin\theta(s), 0)$. Thus

$$\mathbf{T}'(s) = (-\sin\theta(s), \cos\theta(s), 0)(\tfrac{d\theta}{ds})$$

and $\kappa(s) = |\mathbf{T}'(s)| = |d\theta/ds|$. This shows that the *curvature of a plane curve is the rate of change of the angle the tangent vector field makes with the horizontal*(up to sign). This approach to the curvature of plane curves is essentially due to L. Euler (1763).

Having justified the definition of curvature, we shall now develop some machinery to study curvature. This machinery which is called the Frenet-Serret apparatus, is the key to studying the geometry of curves in $\mathbf{R}^3$ and in fact uniquely determines the curve as we will see in Section 1-5.

It is usual in both elementary physic and mathematics to think of a vector as an arrow with head and a base point (or "point of application"). If we imagine at each point $\boldsymbol{\alpha}(s)$ on the curve the set of all vectors whose base point is $\boldsymbol{\alpha}(s)$, then we obtain at each point $\boldsymbol{\alpha}(s)$ a 3-dimensional vector space. From the point of view of geometry, what is a natural basis for these vector space? Certainly, if $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$, then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for these vector space (for each $s$). The problem is that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

reflects the geometry of $\mathbf{R}^3$ instead of the geometry of the curve and is thus unsatisfactory from a geometric viewpoint. Our line of attack is to take the one "geometric" vector we have (the tangent vector field), find another one (the normal vector field) to obtain an "intrinsically geometric" basis. The following definitions are only valid at those points where $\kappa(s) \neq 0$. Note that $\mathbf{T}'/\kappa$ is a unit vector.

DEFINITION 3 The *principal normal vector field* to a unit speed curve $\boldsymbol{\alpha}(s)$ is the (unit) vector field $\mathbf{N}(s) = \mathbf{T}'(s)/\kappa(s)$. The *binormal vector field* to $\boldsymbol{\alpha}(s)$ is $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$. The *torsion* of $\boldsymbol{\alpha}(s)$ is the real-valued function

$$\tau(s) = -\langle \mathbf{B}'(s), \mathbf{N}(s) \rangle.$$

We will give the geometric meaning of torsion in the next section. (It will measure "how far from lying in a plane" $\boldsymbol{\alpha}$ is.)

DEFINITION 4

(a) The Frenet-Serret apparatus of a unit curve $\boldsymbol{\alpha}(s)$ is

$$\{\kappa(s), \tau(s), \mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}.$$

(b) If $\boldsymbol{\beta}(t)$ is a regular curve we may write $t = t(s)$ or $s = s(t)$ by Theorem 2.2. Let $\boldsymbol{\alpha}(s) = \boldsymbol{\beta}(t(s))$ be a unit speed reparameterization of $\boldsymbol{\beta}$ and let $\{\kappa(s), \tau(s), \mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ be the Frenet-Serret apparatus of the unit speed curve $\boldsymbol{\alpha}$. The Frenet-Serret apparatus of $\boldsymbol{\beta}(t)$ is

$$\{\kappa(s(t)), \tau(s(t)), \mathbf{T}(s(t)), \mathbf{N}(s(t)), \mathbf{B}(s(t))\}.$$

Because it is so difficult (and often impossible) to find $t$ as a function of $s$ explicitly, we shall need a computational tool for finding the Frenet-Serret apparatus for a non-unit speed curve. This is done in Section 1-6.

EXAMPLE 3.1 Let $\boldsymbol{\alpha}(s) = (r\cos(s/r), r\sin(s/r), 0)$ be a circle of radius $r > 0$. $\mathbf{T}(s) = (-\sin(s/r), \cos(s/r), 0)$ so that $\kappa(s) = |\mathbf{T}'(s)| = 1/r$, as we mentioned earlier in this section. Note that

$$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\kappa(s)} = \left(-\cos\left(\frac{s}{r}\right), -\sin\left(\frac{s}{r}\right), 0\right)$$

and $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s) = (0, 0, 1)$. Since $\mathbf{B}' = 0$, we see that $\tau(s) = 0$. We have completed the computation of the Frenet-Serret apparatus in this case. We sketch $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ at some points in Figure 1.6.

EXAMPLE 3.2 Consider the unit speed circular helix

$$\boldsymbol{\alpha}(s) = (r\cos\omega s, r\sin\omega s, h\omega s),$$

where $\omega = (r^2 + h^2)^{-1/2}$.

$$\begin{aligned}\mathbf{T}(s) &= \omega(-r\sin\omega s, r\cos\omega s, h)\\ \mathbf{T}'(s) &= -\omega^2 r(\cos\omega s, \sin\omega s, 0).\end{aligned}$$

Thus $\kappa(s) = \omega^2 r$, which is a constant, yet $\boldsymbol{\alpha}$ is *not* a circle.

$$\begin{aligned}\mathbf{N} &= (-\cos\omega s, -\sin\omega s, 0)\\ \mathbf{B} = \mathbf{T}\times\mathbf{N} &= \omega(h\sin\omega s, -h\cos\omega s, r)\\ \mathbf{B}' &= \omega^2 h(\cos\omega s, \sin\omega s, 0).\end{aligned}$$

Thus $\tau(s) = -\langle\mathbf{B}'(s), \mathbf{N}(s)\rangle = \omega^2 h$. We sketch $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ at several points in Figure 1.7. Note that the curvature and torsion are both constant in this example.

LEMMA 3.1 Let $\boldsymbol{\alpha}(s)$ be a unit speed curve. Then for every $s$ such that $\kappa(s) \neq 0$, the set $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is an orthonormal set.

*Proof:* $\mathbf{T}(s)$ is a unit vector, so $\langle\mathbf{T}, \mathbf{T}'\rangle = 0$. Since $\mathbf{T}' = \kappa\mathbf{N}$ and $\kappa \neq 0$, we have $\langle\mathbf{T}, \mathbf{N}\rangle = 0$ and so $\mathbf{T}$ and $\mathbf{N}$ are orthogonal. Meanwhile, $\mathbf{B} = \mathbf{T}\times\mathbf{N}$ implies that $\mathbf{B}$ is orthogonal to both $\mathbf{T}$ and $(N)$. Since $\mathbf{T}$ and $(N)$ are unit vectors, so is $\mathbf{B} = \mathbf{T}\times\mathbf{N}$. □

Because of this lemma, at every point on the curve where $\kappa \neq 0$ we have an orthonormal set of vectors $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ that move and twist as we move along the curve. This is the reason that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is classically called a "moving frame" or "moving trihedron." Look again at Figures 1.6 and 1.7.

If the curve has isolated points at which $\kappa$ is zero, then our construction is not valid. However, if $\kappa$ is zero on an interval, then our intuition demands that the curve be a straight line. This is indeed the case.

PROPOSITION 3.2 Let $\boldsymbol{\alpha}(s)$ be a unit speed curve with $\kappa(s) \equiv 0$ on an interval $[a, b]$. Then the curve segment $\boldsymbol{\alpha} : [a, b] \to \mathbf{R}^3$ is a straight line.

*Proof:* According to the equation of straight line we must produce a vector $\mathbf{v}$ and a point $x_0$ of $\mathbf{R}^3$ such that $\boldsymbol{\alpha}(s) = x_0 + s\mathbf{v}$. Since $|\mathbf{T}'| = \kappa \equiv 0$, $\mathbf{T}$ is constant. Now for any curve, $\boldsymbol{\alpha}(s) = \int_a^s \mathbf{T}(\sigma)d\sigma + \boldsymbol{\alpha}(a)$. Thus, since $\mathbf{T}(\sigma) = \mathbf{T}$ is constant, $\boldsymbol{\alpha}(s) = (s - a)\mathbf{T} + \boldsymbol{\alpha}(a)$. Hence we may let $\mathbf{v} = \mathbf{T}$ and $x_0 = -a\mathbf{T} + \boldsymbol{\alpha}(a)$. □

We note that any unit speed curve can be broken into segments with $\kappa \equiv 0$ on some and $\kappa = 0$ only at the end points of others. By the above proposition, we completely understand the geometry of these segments where $\kappa \equiv 0$. We shall usually consider only the second type of curve and in fact will quite often restrict ourselves to $\boldsymbol{\alpha} : (a, b) \to \mathbf{R}^3$ with $\kappa \neq 0$. This is because at this stage we are interested in the *local* behavior of curves, which means the behavior of the curve near a particular point. (If $\kappa(c) \neq 0$, then $\kappa(s) \neq 0$ for all $s$ near $c$ since $\kappa$ is continuous.)

At an isolated point where $\kappa = 0$ strange things can happen. Let

$$f(t) = \begin{cases} e^{-1/t^2} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Let $\boldsymbol{\alpha}(t)$ be given by

$$\boldsymbol{\alpha}(t) = \begin{cases} (t, f(t), 0) & \text{if } t < 0 \\ (0, 0, 0) & \text{if } t = 0 \\ (t, 0, f(t)) & \text{if } t > 0. \end{cases}$$

This gives an example of $C^\infty$ curve $\boldsymbol{\alpha} : (-\infty, +\infty) \to \mathbf{R}^3$ with the image of $(-\infty, 0]$ lying in one plane and the the image of $[0, +\infty)$ lying in another. Note that $\kappa$ is zero only at one point $\boldsymbol{\alpha}(0)$!

## 1–4. THE FRENET-SERRET THEOREM AND ITS COROLLARIES

Because the Frenet-Serret apparatus has been defined so geometrically we would expect to get a great deal of information from it. This is indeed the case. After a preliminary lemma, we prove the Frenet-Serret Theorem from which we can derive many geometric corollaries. Even though the lemma below belongs to the realm of linear algebra, we shall prove it in detail here because it is so crucial in the proof of the Frenet-Serret Theorem.

LEMMA 4.1 If $E = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is an orthonormal set of $n$ elements of an $n$-dimensional inner product space $V$, then

(a) $E$ is a base of $V$; and

(b) if $\mathbf{v} \in V$, then $\mathbf{v} = \sum_{i=1}^n \langle \mathbf{e}_i, \mathbf{v} \rangle \mathbf{e}_i$.

*Proof:* (a) Because the number of elements in $E$ is the dimension of $V$ we need only prove that $E$ is linearly independent. Let $c^1, c^2, \cdots, c^n$ be real numbers with $\sum c^i \mathbf{e}_i = 0$. Then $0 = \langle \sum c^i \mathbf{e}_i, \mathbf{e}_j \rangle = \sum c^i \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \sum c^i \delta_{ij} = c^j$ for each $j$. Therefore, $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is linearly independent and a basis for $V$.

(b) Since $E$ is a basis, we know that for each $\mathbf{v} \in V$ there are real numbers $v^j$ such that $\mathbf{v} = \sum v^j \mathbf{e}_j$. Therefore,

$$\langle \mathbf{e}_i, \mathbf{v} \rangle = \langle \mathbf{e}_i, \sum v^j \mathbf{e}_j \rangle = \sum v^j \delta_{ij} = v^i,$$

which proves (b). □

The important thing about the above lemma is not that it tells us that $\mathbf{v} \in V$ *can* be expressed as a linear combination of the elements of $E$, but rather that it tells us *how* to express $\mathbf{v}$ as a linear combination of the elements of $E$. TO appreciate this, consider what you must do if you are given an arbitrary basis of $V$ and are asked to write a given vector $\mathbf{v}$ as $\mathbf{v} = \sum a^i \mathbf{u}_i$. You must solve $n$ linear equations in $n$ unknowns which is, in practice, very difficult. If, however, the basis is an orthonormal one ($V$ is an inner product space), then it is easy—you need only compute $\langle \mathbf{u}_j, \mathbf{v} \rangle$ for each $j$.

THEOREM 4.2 (Ferent-Serret) Let $\boldsymbol{\alpha}(s)$ be a unit speed curve with $\kappa \neq 0$ and Frenet-Serret apparatus $\{\kappa, \tau, \mathbf{T}, \mathbf{N}, \mathbf{B}\}$. Then, for each $s$,

$$\begin{array}{llccc} (a) & \mathbf{T}'(s) = & & \kappa(s)\mathbf{N}(s) & \\ (b) & \mathbf{N}'(s) = & -\kappa(s)\mathbf{T}(s) & & +\tau(s)\mathbf{B}(s) \\ (c) & \mathbf{B}'(s) = & & -\tau(s)\mathbf{N}(s) & \end{array}$$

*Proof:* Lemma 3.1 asserts that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is an orthonormal set. Therefore, according to Lemma 4.1, we may write any vector $\mathbf{v}$ as

$$\mathbf{v} = \langle \mathbf{T}, \mathbf{v} \rangle \mathbf{T} + \langle \mathbf{N}, \mathbf{v} \rangle \mathbf{N} + \langle \mathbf{B}, \mathbf{v} \rangle \mathbf{B}. \tag{4.1}$$

(a) $\mathbf{T}' = \kappa \mathbf{N}$ is the definition of $\kappa$ and $\mathbf{N}$.

(b) Since $\mathbf{N}'$ is a vector, we may apply Equation (4.1) with $\mathbf{v} = \mathbf{N}'$. we first compute the coefficient $\langle \mathbf{T}, \mathbf{N}' \rangle$. Differentiating $0 = \langle \mathbf{T}, \mathbf{N} \rangle$, we have $0 = \langle \mathbf{T}', \mathbf{N} \rangle + \langle \mathbf{T}, \mathbf{N}' \rangle = \langle \kappa \mathbf{N}, \mathbf{N} \rangle + \langle \mathbf{T}, \mathbf{N}' \rangle$ so that $\langle \mathbf{T}, \mathbf{N}' \rangle = -\kappa$. Since $\mathbf{N}$ is a unit vector, $\langle \mathbf{N}, \mathbf{N}' \rangle = 0$ and the second coefficient of (4.1) is zero. To compute the third coefficient, $\langle \mathbf{B}, \mathbf{N}' \rangle$, notice that $\langle \mathbf{B}, \mathbf{N} \rangle = 0$ so that by using the definition of $\tau$, $0 = \langle \mathbf{B}, \mathbf{N}' \rangle + \langle \mathbf{B}', \mathbf{N} \rangle = \langle \mathbf{B}, \mathbf{N}' \rangle - \tau$ and $\langle \mathbf{B}, \mathbf{N}' \rangle = \tau$. Putting these three coefficients in (4.1) with $\mathbf{v} = \mathbf{N}'$ yields formula (b).

(c) Formula (c) is obtained by using Equation (4.1) with $\mathbf{v} = \mathbf{B}'$. The first coefficient $\langle \mathbf{T}, \mathbf{B}' \rangle$ is zero because $0 = \langle \mathbf{T}, \mathbf{B} \rangle$ implies

$$0 = \langle \mathbf{T}', \mathbf{B} \rangle + \langle \mathbf{T}, \mathbf{B}' \rangle = \kappa \langle \mathbf{N}, \mathbf{B} \rangle + \langle \mathbf{T}, \mathbf{B}' \rangle = \langle \mathbf{T}, \mathbf{B}' \rangle,$$

since $\langle \mathbf{N}, \mathbf{B} \rangle = 0$. The definition of $\tau$ gives $\langle \mathbf{N}, \mathbf{B}' \rangle = -\tau$. Finally, since $\mathbf{B}$ is a unit vector, the last coefficient $\langle \mathbf{B}, \mathbf{B}' \rangle$ is zero. □

The reason that we have left spaces in the statement of Theorem 4.2 is that it is easy to remember the form of the equation in matrix format:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}.$$

The fact that the matrix is skew symmetric is very important in more abstract differential geometry.

The equations in the above theorem are naturally called the Frenet-Serret equations. They were independently found by Frenet (1847, published in 1852) and Serret (1851). We show how powerful these equations are by drawing some corollaries and leave other applications for the exercises. The first corollary shows that the vanishing of torsion characterizes plane curves with $\kappa \neq 0$.

COROLLARY 4.3 Let $\boldsymbol{\alpha}(s)$ be a unit speed curve with $\kappa \neq 0$. The following are equivalent:

(a) The image of $\boldsymbol{\alpha}$ lies in a plane(more simply, $\boldsymbol{\alpha}$ is a plane curve).
(b) $\mathbf{B}$ is a constant vector.
(c) $\tau(s) = 0$ for all $s$.

*Proof:* The equivalent of (b) and (c) is given by the Frenet-Serret equation $\mathbf{B}' = -\tau\mathbf{N}$. If $\boldsymbol{\alpha}$ is a plane curve, we may assume (by appropriate choice of coordinates in $\mathbf{R}^3$) that $\boldsymbol{\alpha}$ lies in the $(x, y)$ plane. For this case we have already computed the torsion and it is zero, so (a) implies (b).

We now show (b) implies (a). Let $\mathbf{x}_0$ be any point on $\boldsymbol{\alpha}$, say $\mathbf{x}_0 = \boldsymbol{\alpha}(0)$. According to Section 1–4, we must find a vector $\mathbf{v}$ (which does not, of course, depend on $s$) so that $\langle \boldsymbol{\alpha}(s) - \mathbf{x}_0, \mathbf{v} \rangle$ is identically zero. A glance at the Frenet-Serret equations or a close look at Example 3.1 (when $\mathbf{v} = (0, 0, 1)$) suggests that we should let $\mathbf{v}$ be the binormal vector $\mathbf{B}$.

Since $\mathbf{B}$ is a constant vector, $\mathbf{B}' = 0$. Thus

$$\langle \boldsymbol{\alpha}(s) - \mathbf{x}_0, \mathbf{B} \rangle' = \langle \boldsymbol{\alpha}'(s), \mathbf{B} \rangle + \langle \boldsymbol{\alpha}(s) - \mathbf{x}_0, \mathbf{B}' \rangle = \langle \mathbf{T}, \mathbf{B} \rangle = 0$$

and $\langle \boldsymbol{\alpha}(s) - \mathbf{x}_0, \mathbf{B} \rangle$ is therefore constant. But $\langle \boldsymbol{\alpha}(s) - \mathbf{x}_0, \mathbf{B} \rangle = 0$ at $s = 0$, so that this constant is zero and $\boldsymbol{\alpha}(s)$ lies in a plane. □

Note that the proof actually gives the plane in which $\boldsymbol{\alpha}$ lies. It is the plane through $\mathbf{x}_0$ perpendicular to $\mathbf{B}$. This theorem is actually false if the assumption $\kappa \neq 0$ is omitted. (See the last paragraph in Section 1.3)

DEFINITION 1 The *osculating plane* to a unit speed curve $\boldsymbol{\alpha}$ at the point $\boldsymbol{\alpha}(s)$ is the plane through $\boldsymbol{\alpha}(s)$ perpendicular to $\mathbf{B}$ (and hence spanned by $\mathbf{T}$ and $\mathbf{N}$). The *normal plane* of $\boldsymbol{\alpha}$ is the plane perpendicular to $\mathbf{T}$. The *rectifying plane* of $\boldsymbol{\alpha}$ is the plane perpendicular to $\mathbf{N}$.

What is the significance of the osculating plane? If a curve actually lies in a plane ($\tau = 0$), that plane is the osculating plane. More generally, the osculating plane of $\boldsymbol{\alpha}(s)$ is the plane which $\boldsymbol{\alpha}(s)$ is the closest to being in, just as the tangent line at $\boldsymbol{\alpha}(s)$ is the line that $\boldsymbol{\alpha}$ is the closest to being in. ( Note that the tangent line lies in the osculating plane.) The curve twists out of the osculating plane, and $\tau$ measures this twisting or torsion. As $s$ increases, the curve twists toward the side $\mathbf{B}$ points to if $\tau > 0$ and toward the opposite side if $\tau < 0$. This gives geometric meaning to the sign of $\tau$. Also, since $\mathbf{B}' = -\tau\mathbf{N}$ and $\mathbf{N}$ is the normal to the osculating plane, $\tau$ measures how the osculating plane is turning as $s$ increases. This provides the geometric interpretation of $\tau$ which we promised in the last section.

DEFINITION 2 A (general) *helix* is a regular curve $\boldsymbol{\alpha}$ such that for some fixed unit vector $\mathbf{u}$, $\langle \mathbf{T}, \mathbf{u} \rangle$ is constant. $\mathbf{u}$ is called the *axis* of the helix. Intuitively, the helix grows linearly in the direction of axis.

EXAMPLE 4.1 The right circular helix of Example 1.6 is helix with axis $(0, 0, 1)$.

EXAMPLE 4.1 Any regular plane curve is a helix since $\mathbf{B}$ is constant and may serve as $\mathbf{u}$.

COROLLARY 4.4 (Lancret, 1802) Aunit speed curve $\boldsymbol{\alpha}(s)$ with $\kappa \neq 0$ is a helix if and only if there is a constant $c$ such that $\tau = c\kappa$.

*Proof:* Assume $\boldsymbol{\alpha}$ is a helix. Since $\langle \mathbf{T}, \mathbf{u} \rangle$ is a constant, we may write

$$\langle \mathbf{T}, \mathbf{u} \rangle = \cos\theta$$

where $\theta$ is some fixed angle (called the *pitch* of $\boldsymbol{\alpha}$). If $\theta$ is an integer multiple of $\pi$, then $\mathbf{u} = \mathbf{T}$ or $\mathbf{u} = -\mathbf{T}$. In either case this implies that $\kappa = 0$, which is q contradiction. We may therefore assume that $\theta$ is not an integral multiple of $\pi$. The following computation shows that $\mathbf{N}$ is perpendicular to $\mathbf{u}$:

$$0 = \langle \mathbf{T}, \mathbf{u} \rangle' = \langle \mathbf{T}', \mathbf{u} \rangle = \kappa \langle \mathbf{N}, \mathbf{u} \rangle.$$

Hence Lemma 4.1 shows that $\mathbf{u} = \cos\theta\mathbf{T} + \langle \mathbf{B}, \mathbf{u} \rangle$. Since $|\mathbf{u}| = 1$, we may choose the sign of $\theta$ so that $\langle \mathbf{B}, \mathbf{u} \rangle = \sin\theta$. Then

$$0 = \mathbf{u}' = (\cos\theta)\kappa\mathbf{N} - (\sin\theta)\tau\mathbf{N}$$

and $\kappa\cos\theta = \tau\sin\theta$. Since $\theta$ is not integral multiple of $\pi$, we have that $\tau = c\kappa$ where $\theta = \cot\theta$.

We now assume that $\tau = c\kappa$ and show that $\boldsymbol{\alpha}$ is a helix. Motivated by the first half of the proof, we define $\theta$ by $\cot\theta = c$ with $0 < \theta < \pi$ and let $u = \cos\theta\mathbf{T} + \sin\theta\mathbf{B}$. An application of the Frenet-Serret equations shows that $\mathbf{u}' = 0$, so that $\mathbf{u}$ is a constant. Note also that $\langle \mathbf{T}, \mathbf{u} \rangle = \cos\theta$, which is constant, so that $\boldsymbol{\alpha}$ is a helix. $\square$

Although Lancret stated this theorem in 1802, the first proof was given by De Saint Venan in 1845.

One of the beautiful things about the above corollary is its constructive nature. If you are given the constant $c$ such that $\tau = c\kappa$, then you can actually *compute* the axis of the helix by first finding its pitch and using the values of $\mathbf{T}$ and $\mathbf{B}$ at a single point. As an exercise, the reader can show that if both $\kappa$ and $\tau$ are constant, then $\boldsymbol{\alpha}$ is a circular helix.

We end this section with several propositions which illustrate how the Frenet-Serret Theorem can be applied to derive simple geometric results. Many more such results are contained in the exercises. The reader should note that the general method of proving these results is as follows: (1) express the geometric hypotheses as an algebraic equation using linear algebra; (2) differentiate an appropriate expression (possibly several times), using the Frenet-Serret Theorem and the hypotheses; (3) interpret the result geometrically.

PROPOSITION 4.5 $\boldsymbol{\alpha}(s)$ is a straight line if and only if there is a point $\mathbf{x}_0 \in \mathbf{R}^3$ such that every tangent line to $\boldsymbol{\alpha}(s)$ goes through $\mathbf{x}_0$.

*Proof:* If $\boldsymbol{\alpha}(s)$ is a straight line, any point on $\boldsymbol{\alpha}(s)$ may be chosen as $\mathbf{x}_0$ since the image of $\boldsymbol{\alpha}(s)$ is the tangent line at each point.

Now suppose that every tangent line to $\boldsymbol{\alpha}$ goes through $\mathbf{x}_0$. Then

$$\boldsymbol{\alpha}(s) - \mathbf{x}_0 = \lambda(s)\mathbf{T}(s)$$

for some function $\lambda(s)$. Either $\kappa = 0$ or $\mathbf{N}(s)$ is defined. In the second case

$$\mathbf{T}(s) = \boldsymbol{\alpha}'(s) = \lambda'(s)\mathbf{T}(s) + \lambda(s)\mathbf{T}'(s) = \lambda'(s)\mathbf{T}(s) + \lambda(s)\kappa(s)\mathbf{N}(s).$$

Hence $(\lambda'(s) - 1)\mathbf{T} + \lambda(s)\kappa(s)\mathbf{N} = 0$. Since $\mathbf{T}$ and $\mathbf{N}$ are linearly independent, $\lambda'(s) \equiv 1$ and $\lambda(s)\kappa(s) \equiv 0$. Thus $\lambda(s) = s + c$, which is not constant, and hence $\kappa(s) \equiv 0$. Then $\boldsymbol{\alpha}$ is a straight line by Proposition 3.2. □

PROPOSITION 4.6 Let $\boldsymbol{\alpha}(s)$ be a unit speed curve such that every normal plane to $\boldsymbol{\alpha}(s)$ goes through a given fixed pint $\mathbf{x}_0 \in \mathbf{R}^3$. Then the image of $\boldsymbol{\alpha}$ lies on a sphere.

*Proof:* The normal plane is orthogonal to $\mathbf{T}$, so that $\langle \boldsymbol{\alpha}(s) - \mathbf{x}_0, \mathbf{T} \rangle = 0$. Then $\langle \boldsymbol{\alpha} - \mathbf{x}_0, \boldsymbol{\alpha} - \mathbf{x}_0 \rangle = 2\langle \boldsymbol{\alpha} - \mathbf{x}_0, \mathbf{T} \rangle = 0$ and $\langle \boldsymbol{\alpha} - \mathbf{x}_0, \boldsymbol{\alpha} - \mathbf{x}_0 \rangle$ is a constant $a \geq 0$. If $a = 0$, then $\boldsymbol{\alpha}'(s) \equiv \mathbf{x}_0$ and $\boldsymbol{\alpha}(s)$ is not regular. Hence $a > 0$ and $\boldsymbol{\alpha}(s)$ lies on a sphere with center $\mathbf{x}_0$ and radius $\sqrt{a}$. □

PROPOSITION 4.7 Let $\boldsymbol{\alpha}(s)$ be a unit speed curve with $\kappa \neq 0$. Then $\boldsymbol{\alpha}(s)$ lies in a plane if and only if all osculating planes are parallel.

*Proof:* If $\boldsymbol{\alpha}(s)$ lies in a plane, the plane is he osculating plane at each point. Hence all osculating planes are parallel.

Suppose all osculating planes are parallel. Then the values of $\mathbf{B}(s)$ at two points are parallel. Hence $\mathbf{B}(s)$ is constant and $\boldsymbol{\alpha}$ lies in a plane by Proposition 4.3 □

PROPOSITION 4.8 Let $\boldsymbol{\alpha}(s)$ be a unit speed curve whose image lies on a sphere of radius $r$ and center $\mathbf{m}$. Then $\kappa \neq 0$. If $\tau \neq 0$, then

$$\boldsymbol{\alpha} - \mathbf{m} = -\rho\mathbf{N} - \rho'\sigma\mathbf{B},$$

where $\rho = 1/\kappa$(called *radius of curvature*) and $\sigma = 1/\tau$ (called *radius of torsion*). Hence $r^2 = \rho^2 + (\rho'\sigma)^2$.

*Proof:* We have $\langle\boldsymbol{\alpha}(s) - \mathbf{m}, \boldsymbol{\alpha}(s) - \mathbf{m}\rangle = r^2$, so that

$$0 = \langle\boldsymbol{\alpha}(s) - \mathbf{m}, \boldsymbol{\alpha}(s) - \mathbf{m}\rangle' = 2\langle\boldsymbol{\alpha}(s) - \mathbf{m}, \mathbf{T}\rangle.$$

Then

$$\begin{aligned} 0 &= \langle\boldsymbol{\alpha}(s) - \mathbf{m}, \mathbf{T}\rangle' = \langle\mathbf{T}, \mathbf{T}\rangle + \langle\boldsymbol{\alpha}(s) - \mathbf{m}, \mathbf{T}'\rangle \\ &= 1 + \langle\boldsymbol{\alpha}(s) - \mathbf{m}, \kappa\mathbf{N}\rangle, \end{aligned}$$

or $\langle\boldsymbol{\alpha}(s) - \mathbf{m}, \kappa\mathbf{N}\rangle = -1 \neq 0$. Thus $\kappa \neq 0$.

Assume $\tau \neq 0$. $\boldsymbol{\alpha}(s) - \mathbf{m} = a\mathbf{T} + b\mathbf{N} + c\mathbf{B}$, where the coefficients $a, b, c$ may be found by Lemma 4.1.

$$\begin{aligned} a &= \langle\boldsymbol{\alpha}(s) - \mathbf{m}, \mathbf{T}\rangle = 0. \\ b &= \langle\boldsymbol{\alpha}(s) - \mathbf{m}, \mathbf{N}\rangle = -\frac{1}{\kappa} = -\rho. \\ c &= \langle\boldsymbol{\alpha}(s) - \mathbf{m}, \mathbf{B}\rangle. \end{aligned}$$

Since $\langle\boldsymbol{\alpha}(s) - \mathbf{m}, \mathbf{N}\rangle = -\rho$,

$$\begin{aligned} -\rho' &= \langle\boldsymbol{\alpha}(s) - \mathbf{m}, \mathbf{N}\rangle' = \langle\mathbf{T}, \mathbf{N}\rangle + \langle\boldsymbol{\alpha}(s) - \mathbf{m}, -\kappa\mathbf{T} + \tau\mathbf{B}\rangle \\ &= 0 - \kappa\langle\boldsymbol{\alpha}(s) - \mathbf{m}, \mathbf{T}\rangle + \tau\langle\boldsymbol{\alpha}(s) - \mathbf{m}, \mathbf{B}\rangle \\ &= 0 + \tau\langle\boldsymbol{\alpha}(s) - \mathbf{m}, \mathbf{B}\rangle. \end{aligned}$$

Hence $c = \langle\boldsymbol{\alpha}(s) - \mathbf{m}, \mathbf{B}\rangle = -\rho'/\tau = -\rho'\sigma$. Thus

$$\boldsymbol{\alpha}(s) - \mathbf{m} = -\rho\mathbf{N} - \rho'\sigma\mathbf{B}.$$

Since $\mathbf{N}$ and $\mathbf{B}$ are orthonormal,

$$\begin{aligned} r^2 &= \langle\boldsymbol{\alpha}(s) - \mathbf{m}, \boldsymbol{\alpha}(s) - \mathbf{m}\rangle \\ &= |-\rho\mathbf{N} - \rho'\sigma\mathbf{B}|^2 = \rho^2 + (\rho'\sigma)^2. \end{aligned}$$

□

It is possible to generalize much of the previous material to curves in higher dimensional Euclidean spaces—$\boldsymbol{\alpha} : (a, b) \to \mathbf{R}^n$. The interested reader might consult H. Gluck, *Higher curvatures of curves in Euclidean space, I, II*, American Mathematical Monthly, 1966, **73**: 699–704; 1967, **74**: 1049–1056. The Frenet-Serret frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is just one geometrically nice basis for the vector spaces along a curve. There are other possibilities. See R. L. Bishop, *There is more than one way to frame a curve*, American Mathematical Monthly, 1975, **82**: 246–251.

## 1–5. EXISTENCE AND UNIQUENESS THEOREM FOR CURVES

After the Frenet-Serret Theorem we draw several corollaries which characterized several types of curves by properties of their curvature and torsion:

| | |
|---|---|
| $\kappa \equiv 0$ | straight line (Proposition 3.2) |
| $\kappa \neq 0, \tau \equiv 0$ | plane curve (Corollary 4.3) |
| $\dfrac{\kappa}{\tau} = \text{constant}$ | helix (Corollary 4.6) |
| As problems we will have: | |
| $\tau \equiv 0,\ \kappa = \text{constant} > 0$ | circle (Exercise) |
| $\tau = \text{constant} \neq 0, \kappa = \text{constant} > 0$ | circular helix (Exercise) |

There were also a large number of exercises where curvature and torsion played a major role in describing what was occurring geometrically.

This is not at all accidental. Theorem 5.2 will tell us that as long as $\kappa \neq 0$, the functions $\kappa$ and $\tau$ completely describe the curve geometrically, except for its position in space. Philosophically, this is the same as saying that two angles and one side completely describe a triangle in plane geometry, or that the radius completely describe a circle. Classically, mathematicians have called $\kappa = \kappa(s),\ \tau = \tau(s)$ the intrinsic equations of a curve.

The actual proof of Theorem 5.2 will not be used in the sequel and may be omitted. However, the content of the theorem is important and will be used. The proof will depend on the following basic result from the theory of ordinary differential equations which is due, in various formulations, to Picard, Lindelöf, Peano, and Cauchy. The dependence of the Fundamental Theorem of Curves on a major theorem of ordinary differential equation shows that in the differential approach to geometry, the heart of geometry is ultimately in differential equations. We will see this again when we study geodesics and the Fundamental Theorem of Surfaces in Chapter two.

THEOREM 5.1 (Picard) Suppose that the $\mathbf{R}^n$-valued function $\mathbf{A}(\mathbf{x}, t)$ is defined and continuous in the closed region $|\mathbf{x} - \mathbf{c}| \leq K,\ |t - a| \leq T$, and satisfies a Lipschitz condition there. Let $M = \sup|\mathbf{A}(\mathbf{x}, t)|$ over this region. Then the differential equation $d\boldsymbol{\alpha}/dt = \mathbf{A}(\boldsymbol{\alpha}, t)$ has a unique solution on the interval $|t - a| \leq \min(T, K/M)$ satisfying $\boldsymbol{\alpha}(a) = \mathbf{c}$. (The technical requirement about the Lipschitz condition is satisfied if $\mathbf{A}$ has bounded partial derivatives with respect to the coordinates of $\mathbf{R}^n$.)

THEOREM 5.2 (Fundamental Theorem of Curves) Any regular curve with $\kappa > 0$ is completely determined, up to position, by its curvature and torsion. More precisely, let $(a, b)$ be an interval about zero, $\bar{\kappa}(s) > 0$ a $C^1$ function on $(a, b)$, $\bar{\tau}(s)$ a continuous function on $(a, b)$, $\mathbf{x}_0$ a fixed point of $\mathbf{R}^3$, and $\{\mathbf{D}, \mathbf{E}, \mathbf{F}\}$ a fixed right handed orthonormal basis of $\mathbf{R}^3$. Then there exists a unique $C^3$ curve $\boldsymbol{\alpha} : (a, b) \to \mathbf{R}^3$ such that

(a) the parameter is arc length from $\boldsymbol{\alpha}(0)$;
(b) $\boldsymbol{\alpha}(0) = \mathbf{x}_0$, $\mathbf{T}(0) = \mathbf{D}$, $\mathbf{N}(0) = \mathbf{E}$, $\mathbf{B}(0) = \mathbf{F}$; and
(c) $\kappa(s) = \bar{\kappa}(s)$, $\tau(s) = \bar{\tau}(s)$.

*Proof:* Consider the system of ordinary differential equations

$$\mathbf{u}'_j = \sum_{i=1}^{3} a^i_j \mathbf{u}_i, \quad j = 1, 2, 3,$$

where $(a^i_j)$ is the matrix $\begin{pmatrix} 0 & \bar{\kappa} & 0 \\ -\bar{\kappa} & 0 & \bar{\tau} \\ 0 & -\bar{\tau} & 0 \end{pmatrix}$. Picard's Theorem implies that this system has a unique solution $\mathbf{u}_j(s)$ with $\mathbf{u}_1(0) = \mathbf{D}$, $\mathbf{u}_2(0) = \mathbf{E}$, $\mathbf{u}_3(0) = \mathbf{F}$. We shall show that the $\mathbf{u}_i$ give the moving trihedron of a regular $C^3$ space curve $\boldsymbol{\alpha}$ with the required curvature and torsion.

*Step 1.* The vectors $\mathbf{u}_i$ are orthonormal:
*Proof:* Let $p_{ij} = \langle \mathbf{u}_i, \mathbf{u}_j \rangle$ so that

$$p'_{ij} = \langle \mathbf{u}'_i, \mathbf{u}_j \rangle + \langle \mathbf{u}_i, \mathbf{u}'_j \rangle = \sum a^k_i p_{kj} + \sum a^k_j p_{ik}.$$

Thus $p_{ij}$ satisfies the initial value problem

$$p'_{ij} = \sum (a^k_i p_{kj} + a^k_j p_{ik})$$
$$p_{ij}(0) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

By Picard's Theorem, such a system has a unique solution. But

$$\sum (a^k_i \delta_{kj} + a^k_j \delta_{ik}) = a^j_i + a^i_j \equiv 0 \equiv \delta'_{ij}.$$

Hence $\delta_{ij} \equiv p_{ij}$ gives a solution, and thus the only solution: $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \delta_{ij}$ and $\mathbf{u}_i$ are orthonormal.

*Step 2.* Let $\boldsymbol{\alpha}(s) = \mathbf{x}_0 + \int_0^s \mathbf{u}_1(\sigma)d\sigma$ for $s \in (a, b)$. Then $\boldsymbol{\alpha}(s)$ is $C^3$, regular, and unit speed:
*Proof:* $d\boldsymbol{\alpha}/ds = \mathbf{u}_1(s)$ by the fundamental theorem of calculus.

$$\frac{d^2\boldsymbol{\alpha}}{ds^2} = \mathbf{u}_1' = \bar{\kappa}(s)\mathbf{u}_2.$$

Both $\bar{\kappa}$ and $\mathbf{u}_2$ are differentiable.

$$\frac{d^3\boldsymbol{\alpha}}{ds^3} = \bar{\kappa}'\mathbf{u}_2 + \bar{\kappa}\mathbf{u}_2' = \bar{\kappa}'\mathbf{u}_2 + \bar{\kappa}(-\bar{\kappa}\mathbf{u}_1 + \bar{\tau}\mathbf{u}_3).$$

Since $\bar{\kappa}$ is $C^1$, $\bar{\tau}$ is continuous, and the $\mathbf{u}_i$ are differentiable (hence continuous), $d^3\boldsymbol{\alpha}/ds^3$ is continuous, and $\boldsymbol{\alpha}$ is $C^3$. $|d\boldsymbol{\alpha}/ds| = |\mathbf{u}_1| = 1$ by Step (1). Hence $\boldsymbol{\alpha}$ is regular and unit speed.

*Step 3.* $\bar{\kappa} = \kappa,\ \bar{\tau} = \tau,\ \mathbf{u}_1 = \mathbf{T}, \mathbf{u}_2 = \mathbf{N},\ \mathbf{u}_3 = \mathbf{B}$:
*Proof:* $\boldsymbol{\alpha}' = \mathbf{u}_1$ so $\mathbf{u}_1 = \mathbf{T}$. $\kappa\mathbf{N} = \mathbf{T}' = \mathbf{u}_1' = \bar{\kappa}\mathbf{u}_2$. Since both $\mathbf{N}$ and $\mathbf{u}_2$ are unit vectors and $\bar{\kappa} > 0$, $\bar{\kappa} = \kappa$. Hence $\mathbf{u}_2 \equiv \mathbf{N}$ also. Now $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \pm 1$ since the vectors are orthonormal. At $s = 0$ we have

$$(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = (\mathbf{D}, \mathbf{E}, \mathbf{F}) = \pm 1.$$

Since $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is continuous on $(a, b)$, it is always $+1$ and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is right handed. Thus $\mathbf{B} = \mathbf{T} \times \mathbf{N} \equiv \mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3$. Finally,

$$-\tau\mathbf{N} = \mathbf{B}' = \mathbf{u}_3' = \bar{\tau}\mathbf{u}_2,$$

so $\bar{\tau} = \tau$.

This completes the proof of the existence of a curve $\boldsymbol{\alpha}(s)$ with required curvature, torsion, starting point and initial moving trihedron. On the other hand, the definition of $\boldsymbol{\alpha}(s)$ was forced if $\boldsymbol{\alpha}$ was to solve the problem. Hence there is a *unique* curve with the required property. □

In general, given $\kappa$ and $\tau$ it is very difficult to solve the Frenet-Serret equations and find the curve $\boldsymbol{\alpha}$. However, it can (almost) be done in the case of a helix.

EXAMPLE 5.1 Let $\boldsymbol{\alpha}(s)$ be a helix with $\kappa > 0,\ \tau = c\kappa$ for some constant $c$. It will be useful to reparameterize $\boldsymbol{\alpha}$ by a parameter $t$ given by

$$t(s) = \int_0^s \kappa(\sigma)d\sigma.$$

Note that this is an allowable change of coordinates since $t' = \kappa > 0$ implies $t(s)$ is one-to-one and both $t(s)$ and $s(t)$ are differentiable. Since $\tau = c\kappa$, the Frenet-Serret equations are

$$\mathbf{T}' = \kappa\mathbf{N}, \quad \mathbf{N}' = -\kappa\mathbf{T} + c\kappa\mathbf{B}, \quad \mathbf{B}' = -c\kappa\mathbf{N}.$$

In terms of the parameter $t$ they are

$$\frac{d\mathbf{T}}{dt} = \mathbf{N}, \quad \frac{d\mathbf{N}}{dt} = -\mathbf{T} + c\mathbf{B}, \quad \frac{d\mathbf{B}}{dt} = -c\mathbf{N}.$$

Thus $d^2\mathbf{N}/dt^2 = -\mathbf{N} - c^2\mathbf{N} = -\omega^2\mathbf{N}$, where $\omega = \sqrt{1+c^2}$. Since $\mathbf{N}$ solves this differential equation, we have $\mathbf{N} = \cos\omega t\mathbf{a} + \sin\omega t\mathbf{b}$ for some fixed vectors $\mathbf{a}$ and $\mathbf{b}$. $d\mathbf{T}/dt = \mathbf{N}$ may be integrated to give $\mathbf{T} = (\sin\omega t\mathbf{a} - \cos\omega t\mathbf{b})/\omega$. Hence

$$\boldsymbol{\alpha}(s) = \frac{1}{\omega}\left(\int_0^s \sin\omega t(\sigma)d\sigma\,\mathbf{a} - \int_0^s \cos\omega t(\sigma)d\sigma\,\mathbf{b} + s\mathbf{c} + \mathbf{d}\right).$$

However, the integration constants $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are not arbitrary.

$$\frac{d\mathbf{N}}{dt} = -\omega\sin\omega t\mathbf{a} + \omega\cos\omega t\mathbf{b}$$

so that

$$\begin{aligned} 0 &\equiv \left\langle \mathbf{N}, \frac{d\mathbf{N}}{dt}\right\rangle \\ &= (-\omega|\mathbf{a}|^2 + \omega|\mathbf{b}|^2)\sin\omega t\cos\omega t + \omega\langle\mathbf{a},\mathbf{b}\rangle(\cos^2\omega t - \sin^2\omega t). \end{aligned}$$

At $t = 0$ this equation is $\langle\mathbf{a},\mathbf{b}\rangle$. Then

$$\frac{1}{2}(-\omega|\mathbf{a}|^2 + \omega|\mathbf{b}|^2)\sin 2\omega t \equiv 0$$

and $|\mathbf{a}|^2 = |\mathbf{b}|^2$.

$$1 = |\mathbf{N}|^2 = |\mathbf{a}|^2\cos^2\omega t + |\mathbf{b}|^2\sin^2\omega t$$

yields $|\mathbf{a}| = |\mathbf{b}| = 1$. Thus $\mathbf{a}$ and $\mathbf{b}$ are orthonormal.

Similarly $0 = \langle\mathbf{T},\mathbf{N}\rangle$ yields $\langle\mathbf{a},\mathbf{c}\rangle = 0$ and then $\langle\mathbf{b},\mathbf{c}\rangle = 0$. $1 = |\mathbf{T}|^2$ gives $|\mathbf{c}| = |c|$ and so $\mathbf{c} = \pm c(\mathbf{a}\times\mathbf{b})$. $d\mathbf{N}/dt = -\mathbf{T} + c\mathbf{B}$ implies that $c\mathbf{B} = d\mathbf{N}/dt + \mathbf{T}$. Hence

$$\begin{aligned} \frac{c}{\omega}(\sin\omega t\mathbf{a} - \cos\omega t\mathbf{b} + \mathbf{c}) &= c\mathbf{T} = \mathbf{N}\times c\mathbf{B} \\ &= (\cos\omega t\mathbf{a} + \sin\omega t\mathbf{b}) \times \left[\left(\frac{1}{\omega} - \omega\right)(\sin\omega t\mathbf{a} - \cos\omega t\mathbf{b}) + \frac{1}{\omega}\mathbf{c}\right] \\ &= \frac{1}{\omega}[\cos\omega t(\mathbf{a}\times\mathbf{c}) + \sin\omega t(\mathbf{b}\times\mathbf{c}) + c^2(\mathbf{a}\times\mathbf{b})]. \end{aligned}$$

Thus $\mathbf{c} = +c(\mathbf{a} \times \mathbf{b})$. In terms of the orthonormal basis $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ we have

$$\boldsymbol{\alpha}(s) = \frac{1}{\omega}\Big(\int_0^s \sin \omega t(\sigma) d\sigma, -\int_0^s \cos \omega t(\sigma) d\sigma, cs\Big) + \mathbf{d}_1,$$

where $t(\sigma) = \int_0^\sigma \kappa(s) ds$ and $\mathbf{d}_1 = \boldsymbol{\alpha}(0)$.

Note that the above solution requires that you be able to compute some nontrivial integrals, which may not be all that easy to do. See Exercises.

EXAMPLE 5.2 Using Theorem 5.2 we offer an analytic proof of corollary 4.3. We must show that if $\boldsymbol{\alpha}$ is a unit speed curve with $\tau \equiv 0$, then $\boldsymbol{\alpha}$ lies in a plane. In Example 5.3 we may put $c = 0$. Equation (5.1) yields the curve $\boldsymbol{\beta}(s) = (\int_0^s \sin t(\sigma) d\sigma, -\int_0^s \cos t(\sigma) d\sigma, 0)$, where $t(\sigma) = \int_0^\sigma \kappa(s) ds$. This is clearly a plane curve, lying in the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ and has curvature $\kappa(s)$ and torsion 0. Hence, by Theorem 5.2, it is the same as $\boldsymbol{\alpha}(s)$, up to position. Thus $\boldsymbol{\alpha}(s)$ is also a plane curve.

The preceding proof, although lacking the geometric appeal of the first proof, does contain a valuable insight: a unit speed plane curve may be obtained from its curvature by three integrations.

## 1–6. NON-UNIT SPEED CURVES

In this final section we shall determine the Frenet-Serret apparatus for a curve that is not given a unit speed parametrization. As we pointed out in Section 1–2, in practice it may not be possible to reparameterized by arc length. Thus one must find alternative computational techniques.

Let $\boldsymbol{\beta}(t)$ be a regular curve and let $s(t)$ denote the arc length function. Then $\boldsymbol{\beta}(t) = \boldsymbol{\alpha}(s(t))$, where $\boldsymbol{\alpha}(s)$ is $\boldsymbol{\beta}(t)$ reparameterized by arc length. Note that $ds/dt = |d\boldsymbol{\beta}/dt| > 0$. We wish to determine the Frenet-Serret apparatus in terms of the variable $t$.

PROPOSITION 6.1 If $\boldsymbol{\beta}(t)$ is a regular curve in $\mathbf{R}^3$, then

(a) $\mathbf{T} = \frac{d\boldsymbol{\beta}/dt}{|d\boldsymbol{\beta}/dt|}$; (b) $\mathbf{B} = \frac{\frac{d\boldsymbol{\beta}}{dt} \times \frac{d^2\boldsymbol{\beta}}{dt^2}}{\left|\frac{d\boldsymbol{\beta}}{dt} \times \frac{d^2\boldsymbol{\beta}}{dt^2}\right|}$; (c) $\mathbf{N} = \mathbf{B} \times \mathbf{T}$;

(d) $\kappa = \frac{\left|\frac{d\boldsymbol{\beta}}{dt} \times \frac{d^2\boldsymbol{\beta}}{dt^2}\right|}{\left|\frac{d\boldsymbol{\beta}}{dt}\right|^3}$; (e) $\tau = \frac{\left(\frac{d\boldsymbol{\beta}}{dt}, \frac{d^2\boldsymbol{\beta}}{dt^2}, \frac{d^3\boldsymbol{\beta}}{dt^3}\right)}{\left|\frac{d\boldsymbol{\beta}}{dt} \times \frac{d^2\boldsymbol{\beta}}{dt^2}\right|}$.

*Proof:* (a) Since $\boldsymbol{\beta}(t) = \boldsymbol{\alpha}(s(t))$, we have $\frac{d\boldsymbol{\beta}}{dt} = \frac{d\boldsymbol{\alpha}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\mathbf{T}$. Because $\frac{ds}{dt} > 0$, $\frac{ds}{dt} = \left|\frac{d\boldsymbol{\beta}}{dt}\right|$ and $\mathbf{T} = \frac{\frac{d\boldsymbol{\beta}}{dt}}{\frac{ds}{dt}} = \frac{\frac{d\boldsymbol{\beta}}{dt}}{\left|\frac{d\boldsymbol{\beta}}{dt}\right|}$.

(d) and (b) Since $\frac{d\boldsymbol{\beta}}{dt} = \frac{ds}{dt}\mathbf{T}$, then

$$\frac{d^2\boldsymbol{\beta}}{dt^2} = \frac{d^2 s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} = \frac{d^2 s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2 \mathbf{T}' = \frac{d^2 s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2 \mathbf{N}.$$

Notice that $\frac{d\boldsymbol{\beta}}{dt}$ is the velocity, this gives the acceleration in terms of its tangential and normal components. Hence

$$\frac{d\boldsymbol{\beta}}{dt} \times \frac{d^2\boldsymbol{\beta}}{dt^2} = \frac{ds}{dt}\mathbf{T} \times \left(\frac{d^2 s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2 \mathbf{N}\right) = \kappa\left(\frac{ds}{dt}\right)^3 \mathbf{B},$$

which implies formula (d) and (b).

(c) Since $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a right handed orthonormal basis, $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.

(e) By a direct computation, we have

$$\begin{aligned}
\frac{d^3\boldsymbol{\beta}}{dt^3} &= \frac{d^3 s}{dt^3}\mathbf{T} + \frac{d^2 s}{dt^2}\frac{d\mathbf{T}}{dt} + \frac{d}{dt}\left(\kappa\left(\frac{ds}{dt}\right)^2\right)\mathbf{N} + \kappa\left(\frac{ds}{dt}\right)^2\frac{d\mathbf{N}}{dt} \\
&= \frac{d^3 s}{dt^3}\mathbf{T} + \frac{ds}{dt}\frac{d^2 s}{dt^2}\mathbf{T}' + \frac{d}{dt}\left(\kappa\left(\frac{ds}{dt}\right)^2\right)\mathbf{N} + \kappa\left(\frac{ds}{dt}\right)^3\mathbf{N}' \\
&= \frac{d^3 s}{dt^3}\mathbf{T} + \kappa\frac{ds}{dt}\frac{d^2 s}{dt^2}\mathbf{N} + \frac{d}{dt}\left(\kappa\left(\frac{ds}{dt}\right)^2\right)\mathbf{N} - \kappa^2\left(\frac{ds}{dt}\right)^3\mathbf{T} + \kappa\tau\left(\frac{ds}{dt}\right)^3\mathbf{B} \\
&= \left(\frac{d^3 s}{dt^3} - \kappa^2\left(\frac{ds}{dt}\right)^3\right)\mathbf{T} + \left(\kappa\frac{ds}{dt}\frac{d^2 s}{dt^2} + \frac{d}{dt}\left(\kappa\left(\frac{ds}{dt}\right)^2\right)\right)\mathbf{N} + \kappa\tau\left(\frac{ds}{dt}\right)^3\mathbf{B}.
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\frac{d\boldsymbol{\beta}}{dt}, \frac{d^2\boldsymbol{\beta}}{dt^2}, \frac{d^3\boldsymbol{\beta}}{dt^3}\right) &= \left\langle \frac{d\boldsymbol{\beta}}{dt} \times \frac{d^2\boldsymbol{\beta}}{dt^2}, \frac{d^3\boldsymbol{\beta}}{dt^3}\right\rangle \\
&= \left\langle \kappa\left(\frac{ds}{dt}\right)^3\mathbf{B}, \frac{d^3\boldsymbol{\beta}}{dt^3}\right\rangle \\
&= \tau\left(\kappa\left(\frac{ds}{dt}\right)^3\right)^2 = \tau\left|\frac{d\boldsymbol{\beta}}{dt} \times \frac{d^2\boldsymbol{\beta}}{dt^2}\right|^2,
\end{aligned}$$

which implies the desired formula. □

For a non-unit speed curve $\boldsymbol{\beta}(t)$, the Frenet-Serret apparatus is a function of $t$, not $s$, and does not satisfy the Frenet-Serret equations. However, we do have the following variation.

Proposition 6.2 Let $\boldsymbol{\beta}(t)$ be a regular curve in $\mathbf{R}^3$, and let $v(t) = \left|\frac{d\boldsymbol{\beta}}{dt}\right|$. Then

$$\begin{aligned}
(a)\quad d\mathbf{T}/dt &= \qquad\qquad \kappa v\mathbf{N} \\
(b)\quad d\mathbf{N}/dt &= -\kappa v\mathbf{T} \qquad\qquad +\tau v\mathbf{B} \\
(c)\quad d\mathbf{B}/dt &= \qquad\quad -\tau v\mathbf{N}.
\end{aligned}$$

# Appendix A

# Linear Algebra and Calculus

Differential geometry has two primary tools: linear algebra and calculus. In this text we assume that the reader is familiar with both. We shall recall the basic facts of linear algebra and calculus and urge the reader to review with care anything which is unfamiliar.

## A–1. VECTOR SPACES

Definition 1.1 A *real vector space* is a set $V$, whose elements are called *vectors*, together with two binary operations $+ : V \times V \to V$ and $\cdot : \mathbf{R} \times V \to V$, called *addition* and *multiplication*, which satisfy the following eight axioms for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and $r, s \in \mathbf{R}$:

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;
(b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$;
(c) there is an element $\mathbf{0}$ in $V$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$;
(d) $(rs) \cdot \mathbf{u} = r \cdot (s \cdot \mathbf{u})$;
(e) $(r + s) \cdot \mathbf{u} = r \cdot \mathbf{u} + s \cdot \mathbf{u}$;
(f) $r \cdot (\mathbf{u} + \mathbf{v}) = r \cdot \mathbf{u} + r \cdot \mathbf{v}$;
(g) $0 \cdot \mathbf{u} = \mathbf{0}$; and
(h) $1 \cdot \mathbf{u} = \mathbf{u}$.

We shall normally omit the multiplication symbol "$\cdot$".

Definition 1.2 A set $\{\mathbf{v}_i \mid i \in I\} \subset V$ is *linearly independent* if whenever a finite linear combination $\sum a^i \mathbf{v}_i$ is zero, then each $a^i$ must also be zero. If it is possible to find s finite linear combination $\sum a^i \mathbf{v}_i = \mathbf{0}$ with some $a^k \neq 0$, then the set is *linear dependent.*

Definition 1.3 A subset $S \subset V$ *spans* $V$ if for each vector $\mathbf{v} \in V$ there are vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ in $S$ and real numbers $a^1, a^2, \cdots, a^r$ such that $\mathbf{v} = \sum a^i \mathbf{v}_i$. (The number of elements used ("$r$") may depend on $\mathbf{v}$.)

Definition 1.4 A *basis* of a vector space $V$ is a linearly independent spanning set.

Theorem 1.1 If $V$ is a vector space, then $V$ has a basis. Any two bases have the same number of elements, or all have infinitely many elements. This number is called the *dimension* of $V$.

If $\{\mathbf{v}_i \mid i \in I\}$ is a basis of $V$, then every vector $\mathbf{v} \in V$ can be uniquely written as a finite sum $\mathbf{v} = \sum a^i \mathbf{v}_i$. The numbers $a^i$ are called the components of $\mathbf{v}$ with respect to the given basis.

Definition 1.5 An *inner product* on a vector space $V$ is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbf{R}$ such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $r, s \in \mathbf{R}$:
(a) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$;
(b) $\langle \mathbf{u}, r\mathbf{v} + s\mathbf{w} \rangle = r\langle \mathbf{u}, \mathbf{v} \rangle + s\langle \mathbf{u}, \mathbf{w} \rangle$; and
(c) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

Example 1.1 Ordinary three-dimensional space, $\mathbf{R}^3$, is a real vector space. It has dimension 3. $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis. The ordinary dot product is an inner product:

$$\langle (a^1, a^2, a^3), (b^1, b^2, b^3) \rangle = a^1 b^1 + a^2 b^2 + a^3 b^3.$$

Example 1.2 The set of all polynomials with real coefficients, $\mathbf{R}[x]$, is a real vector space. $\mathbf{R}[x]$ has infinite dimension. $\{1, x^2, x^3, \cdots, x^n, \cdots\}$ is a basis. We may set $\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)dx$.

Definition 1.6 If $V$ has an inner product and $\mathbf{v} \in V$, the *length* of $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

Lemma 1.2 (Cauchy-Schwarz Inequality) If $\mathbf{u}, \mathbf{v} \in V$, then $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq |\mathbf{u}||\mathbf{v}|$. Furthermore, $|\langle \mathbf{u}, \mathbf{v} \rangle| = |\mathbf{u}||\mathbf{v}|$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

This lemma tells us that $-1 \leq \langle \mathbf{u}, \mathbf{v} \rangle / |\mathbf{u}||\mathbf{v}| \leq 1$ (unless $|\mathbf{u}||\mathbf{v}| = 0$, in which case $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ also.) Hence an angle $\theta$ may be defined by the formula

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{|\mathbf{u}||\mathbf{v}|}.$$

$\theta$, which is defined only up to sign, will be called the *angle between* $\mathbf{u}$ *and* $\mathbf{v}$. If the ordinary dot product is used in $\mathbf{R}^2$, then this concept of an angle coincides with the usual notion of an angle in the plane. Note that

$$\langle \mathbf{u}, \mathbf{v} \rangle = |\mathbf{u}||\mathbf{v}| \cos\theta$$

holds if one of the vectors is zero, even though $\theta$ is not defined then.

DEFINITION 1.7 $\mathbf{u}$ is *orthogonal* (or *perpendicular*) to $\mathbf{v}$ if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

THEOREM 1.3 If $V$ has dimension $n$ and an inner product, then there exist a basis $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ such that $|\mathbf{v}_i| = 1$ and $\mathbf{v}_i$ is perpendicular to $\mathbf{v}_j$ whenever $i \neq j$. Such a basis is called *orthonormal.*

This is proved by showing how to create an orthonormal basis from any given basis by a process called the Gram-Schmidt orthogonalization.

Suppose $V$ has a basis $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ and an inner product $\langle \cdot, \cdot \rangle$. This inner product may be associated with a matrix in the following manner. Let $g_{ij} = \langle u_i, \mathbf{u}_j \rangle$. $(g_{ij})$ is a positive definite symmetric matrix which represents $\langle \cdot, \cdot \rangle$ with respect to the given basis. If $\mathbf{u} = \sum a^i \mathbf{u}_i$ and $\mathbf{v} = \sum b^j \mathbf{u}_j$, then $\langle \mathbf{u}, \mathbf{v} \rangle = \sum a^i b^j g_{ij}$.

Note that if $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is orthonormal, then $(g_{ij}) = (\delta_{ij})$ is the identity matrix. The Kronecker symbol $\delta_{ij}$ (and its variations $\delta_i^j, \delta_j^i, \delta^{ij}$) is defined by

$$\delta = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} \tag{A1.1}$$

If $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is another basis of $V$, we have $\mathbf{u}_i = \sum a_i^\alpha \mathbf{v}_\alpha$. The matrix representing $\langle \cdot, \cdot \rangle$ with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is given by $\bar{g}_{\alpha\beta} = \langle \mathbf{v}_\alpha, \mathbf{v}_\beta \rangle$. Then $g_{ij} = \langle \mathbf{u}_i, \mathbf{u}_j \rangle = \sum a_i^\alpha a_j^\beta \bar{g}_{\alpha\beta}$ or

$$(g_{ij}) = (a_i^\alpha)^t (\bar{g}_{\alpha\beta})(a_j^\beta). \tag{A1.2}$$

This equation shows the effect of a change of basis on the matrix which represents the inner product.

## A–2. LINEAR TRANSFORMATION AND EIGENVECTORS

DEFINITION 2.1 A *linear transformation* is a function $T : V \to W$ of a vector spaces such that $T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$ for all $a, b \in \mathbf{R}$ and $\mathbf{v}, \mathbf{w} \in V$. An *isomorphism* is a one-to-one onto linear transformation.

If $T : V \to W$ is a linear transformation, we may associate a matrix with it. Let $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be a basis of $V$ and let $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\}$ be a basis of $W$. Then there are $mn$ real numbers $T_j^i$ such that $T(\mathbf{v}_j) = \sum T_j^i \mathbf{w}_i$. We say that $(T_j^i)$ represents $T$ with respect to the given bases. (If $V = W$, it is customary to use the same basis for $V$ and $W$). If $\mathbf{v} = \sum a^j \mathbf{v}_j \in V$, then

$$T(\mathbf{v}) = \sum (\sum T_j^i a^j) \mathbf{w}_i.$$

Thus if we view $\mathbf{v}$ as a column vector, the $i$ in $T_j^i$ is the row index and the $j$ is the column index.

Suppose $T : V \to V$ is a linear transformation and $V$ has two bases $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ related by $\mathbf{u}_i = \sum a_i^\alpha \mathbf{v}_\alpha$. If $(T_j^i)$ represents $T$ with respect to the $\mathbf{u}_i$ and $(\bar{T}_\beta^\alpha)$ represents $T$ with respect to the $\mathbf{v}_\alpha$, we have $\sum T_j^i a_i^\alpha \mathbf{v}_\alpha = \sum T_j^i \mathbf{u}_i = T(\mathbf{u}_j) = T(\sum a_j^\beta \mathbf{v}_\beta) = \sum a_j^\beta T(\mathbf{v}_\beta) = \sum a_j^\beta \bar{T}_\beta^\alpha \mathbf{v}_\alpha$. Hence $\sum T_j^i a_i^\alpha = \sum a_j^\beta \bar{T}_\beta^\alpha$, or $(a_i^\alpha)(T_j^i) = (\bar{T}_\beta^\alpha)(a_j^\beta)$, or

$$(T_j^i) = (a_i^\alpha)^{-1}(\bar{T}_\beta^\alpha)(a_j^\beta). \tag{A2.1}$$

This equation shows the effect of a change of basis on the matrix which represents a linear transformation.

Definition 2.2 Let $T : V \to V$ be a linear transformation. A real number $\lambda$ is an *eigenvalue* of $T$ if there is a nonzero vector $\mathbf{v}$ such that $T(\mathbf{v}) = \lambda \mathbf{v}$. $\mathbf{v}$ is called an *eigenvector* of $T$ corresponding to $\lambda$.

If $(T_j^i)$ represents $T$, the eigenvalues of $T$ are real solutions of the polynomial equation $\det(T_j^i - \lambda \delta_j^i) = 0$. Then if the dimension of $V$ is $n$, there are at most $n$ eigenvalues (counting multiplicity). There may be fewer, since some of the solutions might be complex but not real. Once the eigenvalues are known, the eigenvectors are found by solving appropriate linear equation.

Example 2.1 Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be represents with respect to the standard basis $\{(1,0),(0,1)\}$ by the matrix $\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$.

$$\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = (\lambda - 5)(\lambda + 5).$$

The eigenvalues are therefore 5 and $-5$. $\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$ has $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ as one solution. $\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix}$ has $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ as one solution. Hence $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to 5 and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is one corresponding to $-5$.

## A–3. ORIENTATION AND CROSS PRODUCT

Definition 3.1 Let $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be two *ordered* bases of $V$ and define a matrix $(a_j^i)$ by $\mathbf{v}_j = \sum a_j^i \mathbf{u}_i$. If $\det(a_j^i) > 0$, then we say that the ordered bases $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ *give the same orientation to* $V$. They give *opposite orientation* if $\det(a_j^i) < 0$.

The basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ of $\mathbf{R}^3$ will be denoted $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Its orientation will be called *right handed.*

DEFINITION 3.2 If $\mathbf{u}=\sum a^i\mathbf{e}_i$ and $\mathbf{v}=\sum b^i\mathbf{e}_i$ are vectors in $\mathbf{R}^3$, the *cross* (or *vector*) *product* of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u}\times\mathbf{v}=(a^2b^3-a^3b^2)\mathbf{e}_1+(a^3b^1-a^1b^3)\mathbf{e}_2+(a^1b^2-b^2b^1)\mathbf{e}_3.$$

By abuse of notation this may be written as

$$\mathbf{u}\times\mathbf{v}=\det\begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ a^1 & a^2 & a^3\\ b^1 & b^2 & b^3\end{pmatrix}.$$

LEMMA 3.1 Let $\mathbf{u},\mathbf{v}\in\mathbf{R}^3$ and $r\in\mathbf{R}$. Then

(a) $\mathbf{u}\times\mathbf{v}=-\mathbf{v}\times\mathbf{u}$;

(b) $(r\mathbf{u})\times\mathbf{v}=r(\mathbf{u}\times\mathbf{v})$;

(c) $\mathbf{u}\times v=\mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are dependent;

(d) $(\mathbf{u}+\mathbf{v})\times\mathbf{w}=\mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}$;

(e) $\mathbf{u}\times\mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ under the usual dot product of $\mathbf{R}^3$;

(f) $|\mathbf{u}\times\mathbf{v}|=|\mathbf{u}||\mathbf{v}|\sin\theta$, where $\theta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$;

(g) $\{\mathbf{u},\mathbf{v},\mathbf{u}\times\mathbf{v}\}$ gives a right handed orientation to $\mathbf{R}^3$ if $\{\mathbf{u},\mathbf{v}\}$ is linearly independent.

LEMMA 3.2 Let $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{R}^3$. Then $\langle(\mathbf{u}\times\mathbf{v}),\mathbf{w}\rangle=\langle\mathbf{u},(\mathbf{v}\times\mathbf{w})\rangle$.

DEFINITION 3.3 The *mixed* (or *triple*) *scalar product* of $\mathbf{u},\mathbf{v},\mathbf{w}$ is

$$[\mathbf{u},\mathbf{v},\mathbf{w}]\equiv\langle(\mathbf{u}\times\mathbf{v}),\mathbf{w}\rangle.$$

It is important to remember that $[\mathbf{u},\mathbf{v},\mathbf{w}]$ is always a number and *not* a vector.

LEMMA 3.3 $|[\mathbf{u},\mathbf{v},\mathbf{w}]|$ is the volume of the parallelopiped spanned by $\mathbf{u},\mathbf{v},\mathbf{w}$.

## A–4. LINES, PLANES, AND SPHERES

We shall recall the vector equations of lines, planes, and spheres in $\mathbf{R}^3$. Geometrically, a straight line is determined by a point on the line and a vector parallel to it. A plane is determined by a point on the plane and a vector perpendicular to the plane. A sphere is determined by its center and its radius.

DEFINITION 4.1 The *line* through $\mathbf{x}_0$ and parallel to $\mathbf{v} \neq \mathbf{0}$ has equation $\boldsymbol{\alpha}(t) = \mathbf{x}_0 + t\mathbf{v}$.

If we write $\mathbf{x}_0 = (x_0, y_0, z_0)$, $\mathbf{v} = (v^1, v^2, v^3)$ and $\boldsymbol{\alpha}(t) = (x(t), y(t), z(t))$, then the definition gives

$$x(t) - x_0 = tv^1, \quad y(t) - y_0 = tv^2, \quad z(t) - z_0 = tv^3.$$

Assuming that $v^i \neq 0$ for $i = 1, 2, 3$, these equations yield the classical definition of a straight line after solving each equality for $t$ and setting these quantities equal:

$$\frac{x - x_0}{v^1} = \frac{y - y_0}{v^2} = \frac{z - z_0}{v^3},$$

where we have suppressed the $t$ from the notation as is common classically.

If $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct points in $\mathbf{R}^3$ both of which lie on a line $l$, then the vector $\mathbf{x}_2 - \mathbf{x}_1$ is parallel to $l$. This observation proves

LEMMA 4.1 The line through $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbf{R}^3$ has equation

$$\boldsymbol{\alpha}(t) = \mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1).$$

DEFINITION 4.2 The *plane* through $\mathbf{x}_0$ perpendicular to $\mathbf{n} \neq \mathbf{0}$ has equation

$$\langle \mathbf{x} - \mathbf{x}_0, \mathbf{n} \rangle = 0.$$

The following lemma is clear since $\mathbf{u} \times \mathbf{v}$ is perpendicular to the desired plane.

LEMMA 4.2 If $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, the plane through $\mathbf{x}_0$ parallel to both $\mathbf{u}$ and $\mathbf{v}$ has equation $\langle \mathbf{x} - \mathbf{x}_0, \mathbf{u} \times \mathbf{v} \rangle = 0$.

DEFINITION 4.3 The *sphere* in $\mathbf{R}^3$ with center $\mathbf{m}$ and radius $r > 0$ has equation $\langle \mathbf{x} - \mathbf{m}, \mathbf{x} - \mathbf{m} \rangle = r^2$.

If $\mathbf{m} = (a, b, c)$ and $\mathbf{x} = (x, y, z)$, then the definition is the familiar formula

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

## A–5. VECTOR CALCULUS

If abstract vector spaces are unfamiliar, the reader may assume that $V = \mathbf{R}^3$ below. Let $\mathbf{f} : \mathbf{R} \to V$, where $V$ is a real vector space. If $\{\mathbf{v}_1, \mathbf{v_2}, \cdots, \mathbf{v}_n\}$ is a basis of $V$, then $\mathbf{f}(t) = \sum f^i(t)\mathbf{v}_i$. If the component functions $f^i(t)$ are differentiable or integrable, we may differentiate or integrate $\mathbf{f}$ componentwise:

$$\frac{d\mathbf{f}}{dt} = \sum \frac{df^i}{dt}\mathbf{v}_i$$

and

$$\int_a^b \mathbf{f}(t) = \sum \Big( \int_a^b f^i(t)dt \Big)\mathbf{v}_i.$$

We should check that these definitions do not depend on the choice of basis of $V$. However, this is a simple consequence of the linear properties of $d(\quad)/dt$ and $\int_a^b (\quad)dt$ and left to the reader.

Similarly, if $\mathbf{f}$ is a vector-valued function of several variables, we may take partial derivatives or multiple integrals.

LEMMA 5.1 Let $\mathbf{f}, \mathbf{g} : \mathbf{R} \to V$ and suppose that $V$ has an inner product $\langle \cdot, \cdot \rangle$. Then

$$\frac{d}{dt}\langle \mathbf{f}, \mathbf{g} \rangle = \left\langle \frac{d\mathbf{f}}{dt}, \mathbf{g} \right\rangle + \left\langle \mathbf{f}, \frac{d\mathbf{g}}{dt} \right\rangle$$

and

$$\frac{d}{dt}(\mathbf{f} \times \mathbf{g}) = \frac{d\mathbf{f}}{dt} \times \mathbf{g} + \mathbf{f} \times \frac{d\mathbf{g}}{dt}.$$

DEFINITION 5.1 $f : \mathbf{R} \to \mathbf{R}$ is of class $C^k$ if all derivatives up through order $k$ exist and are continuous. $f : \mathbf{R}^n \to \mathbf{R}$ is of class $C^k$ if all its (mixed) partial derivatives of order $k$ or less exist and are continuous. A vector-valued function is of class $C^k$ if all its components with respect to a given basis are of class $C^k$.

Note that if $f$ is of class $C^k$ it is also of class $C^{k-1}$.

Rather than continually worrying about what class a differentiable function belongs to, we shall usually assume it is of class $C^3$. We shall point out those cases where higher class is needed or lower class is sufficient.

Finally, before we start our study of curves, we want to remind you what form the chain rule for differentiation takes. Suppose that $\mathbf{x}$ is a function of several variables $u^1, u^2, \cdots, u^n$ and that the $u^i$ are functions of variables $v^1, v^2, \cdots, v^m$. Then

$$\frac{\partial \mathbf{x}}{\partial v^\alpha} = \sum_{i=1}^n \frac{\partial \mathbf{x}}{\partial u^i}\frac{\partial u^i}{\partial v^\alpha}, \quad \alpha = 1, 2, \cdots, m. \qquad (A5.1)$$

Note that we are writing the coefficients on the right of the vectors instead of the left as would be usual in linear algebra. This is done so that equations (A5.1) look more like the chain rule from calculus.

Special cases that we shall often use arise when $n = m = 2$ as in Equation (A5.2), or $n = 2$ and $m = 1$, as in Equation (A5.3).

$$\frac{\partial \mathbf{x}}{\partial v^\alpha} = \frac{\partial \mathbf{x}}{\partial u^1}\frac{\partial u^1}{\partial v^\alpha} + \frac{\partial \mathbf{x}}{\partial u^2}\frac{\partial u^2}{\partial v^\alpha}, \quad \alpha = 1, 2 \qquad (A5.1)$$

$$\frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{x}}{\partial u^1}\frac{du^1}{dt} + \frac{\partial \mathbf{x}}{\partial u^2}\frac{du^2}{dt}. \qquad (A5.2)$$