Non-linear systems

Job van Amerongen

Control Engineering, Dept. of Electrical Engineering
University of Twente, Netherlands
www.ce.utwente.nl/amn
J.vanAmerongen@utwente.nl

Contents

- Various non-linearities
- reasons for the presence of non-linear elements
- analysis in the phase plane
- analysis with describing functions

Examples of non linearities

- Friction in a mechanical system
- Saturation in an amplifier
- Switching elements, e.g. in thermostats
- operating-point-dependent parameters

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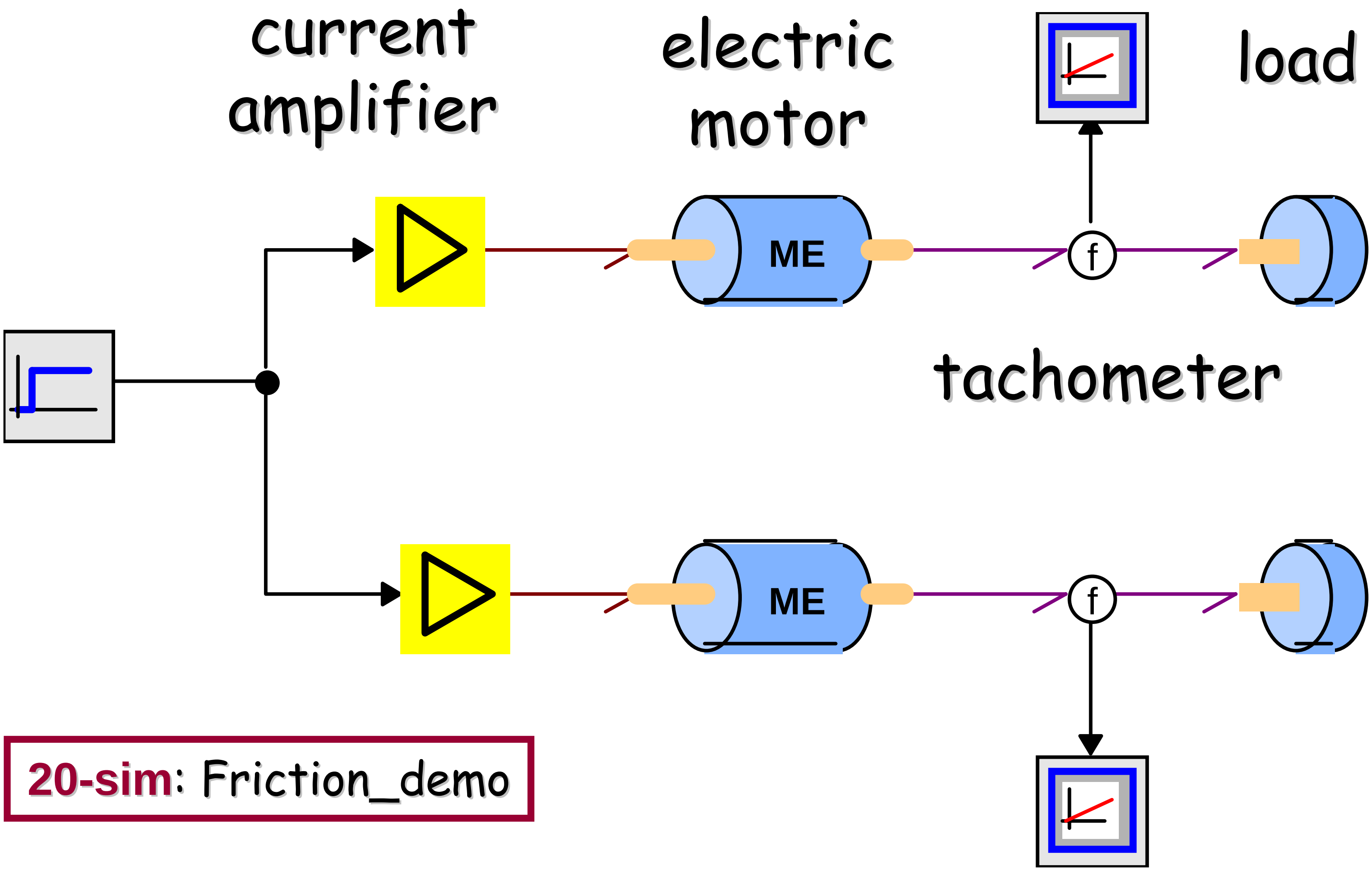
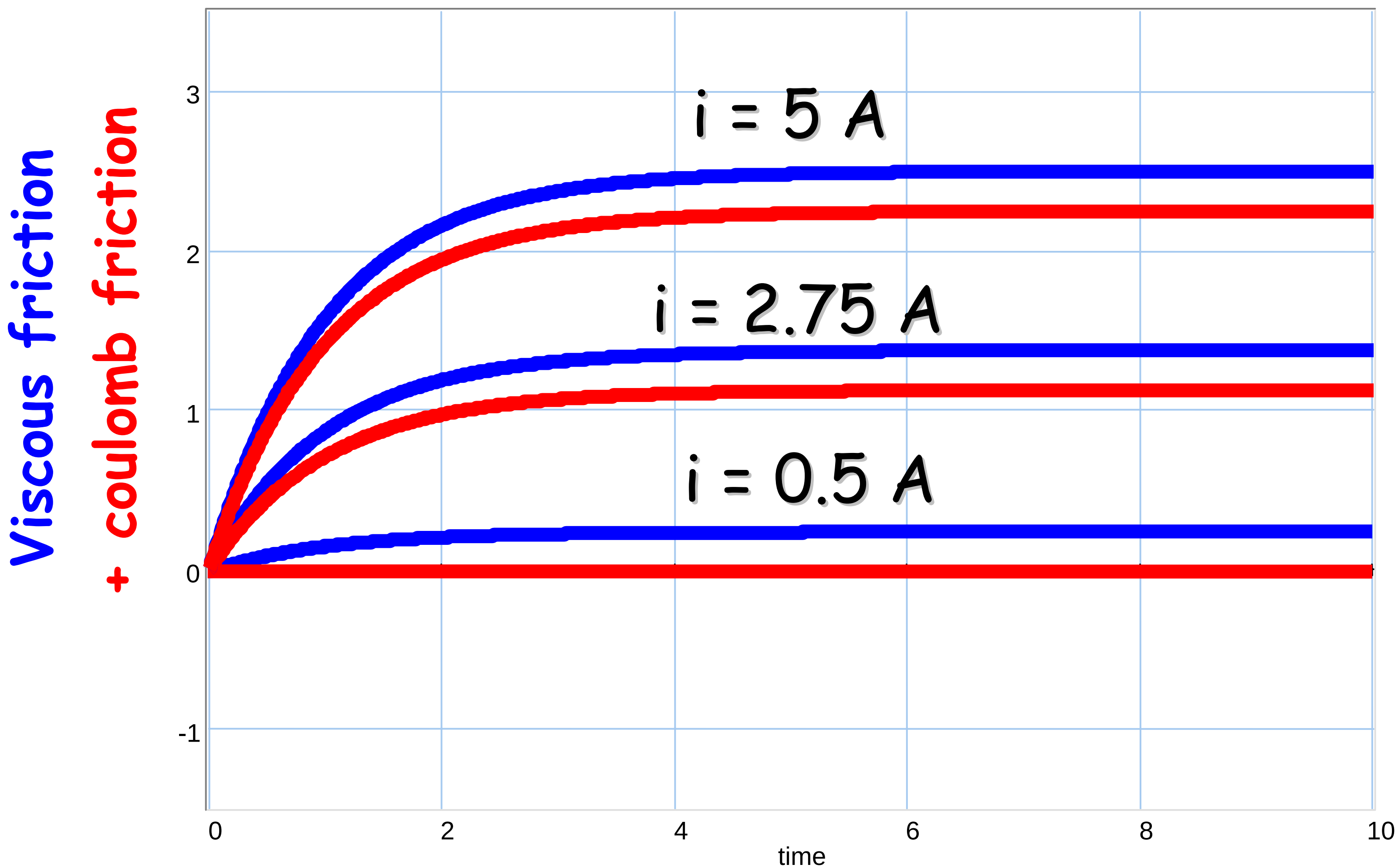
Mechanical friction



20-sim: Friction_demo

Responses



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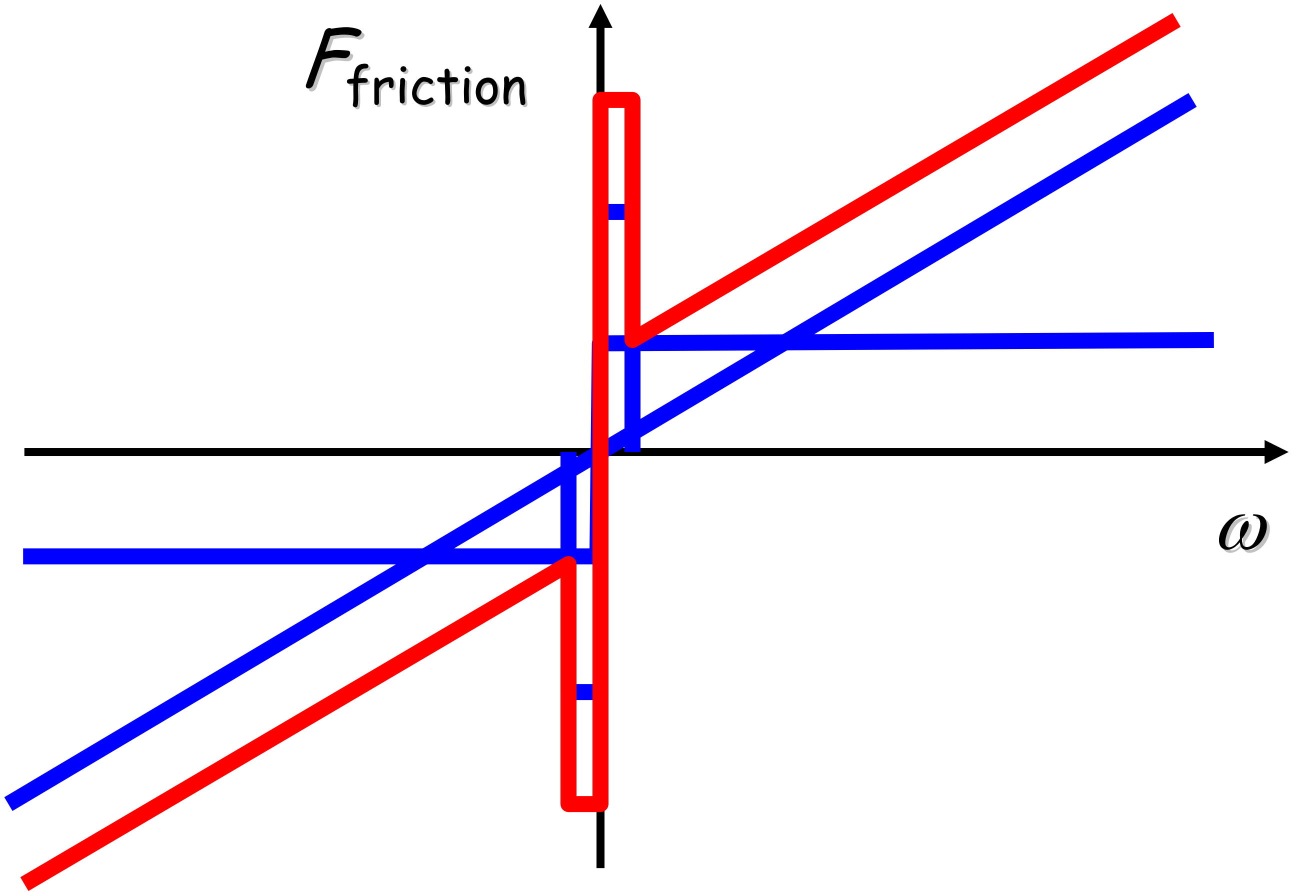
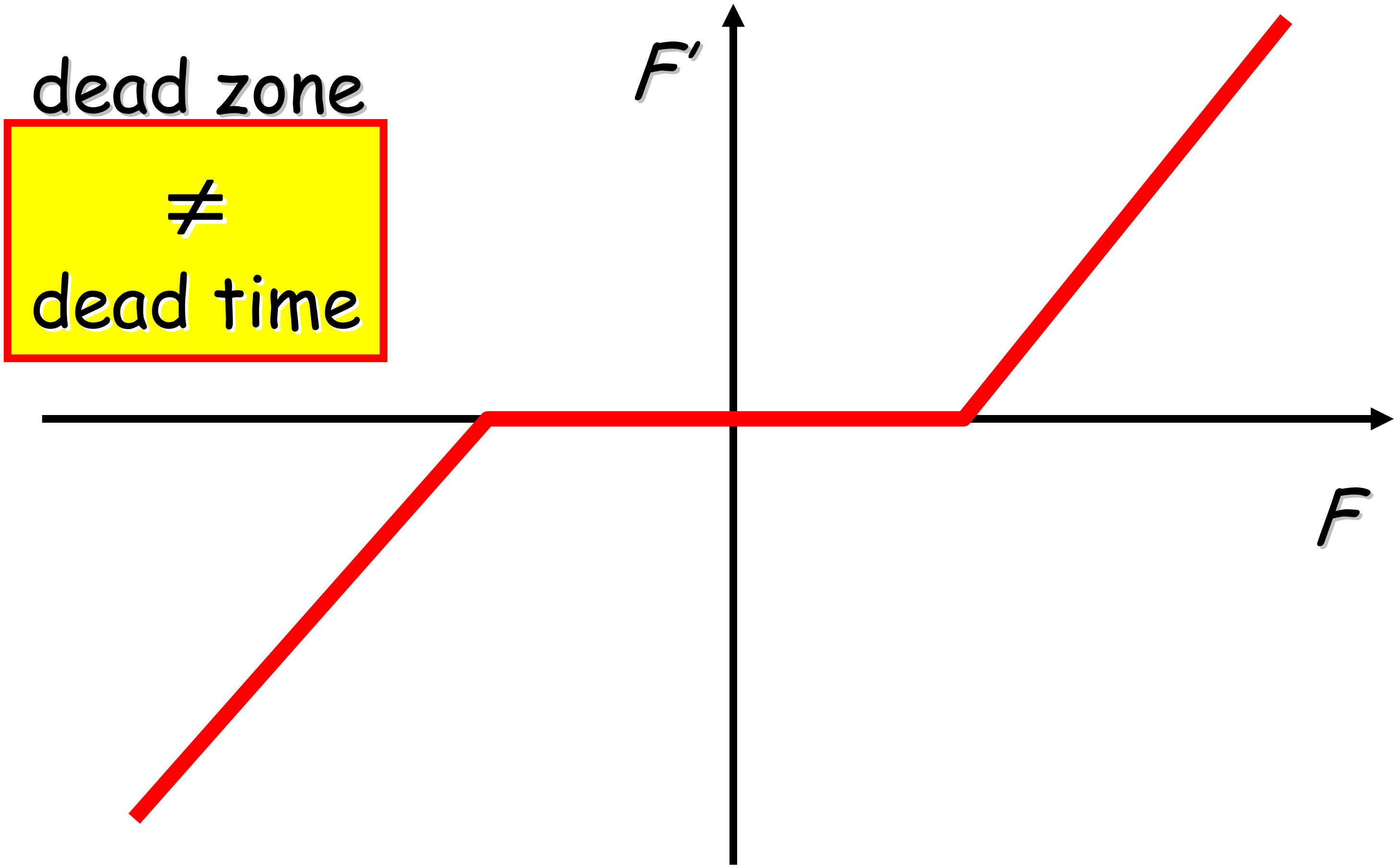
We observe

- For a non-linear system the output can be zero for small input changes
- If the input signal increases with a factor α, the change in the output can be smaller or larger than α.

Friction characteristic



Forward path



Saturation

- Due to
  - maximum valve opening reached
  - output voltage of an amplifier limited by voltage of power supply
  - end stop

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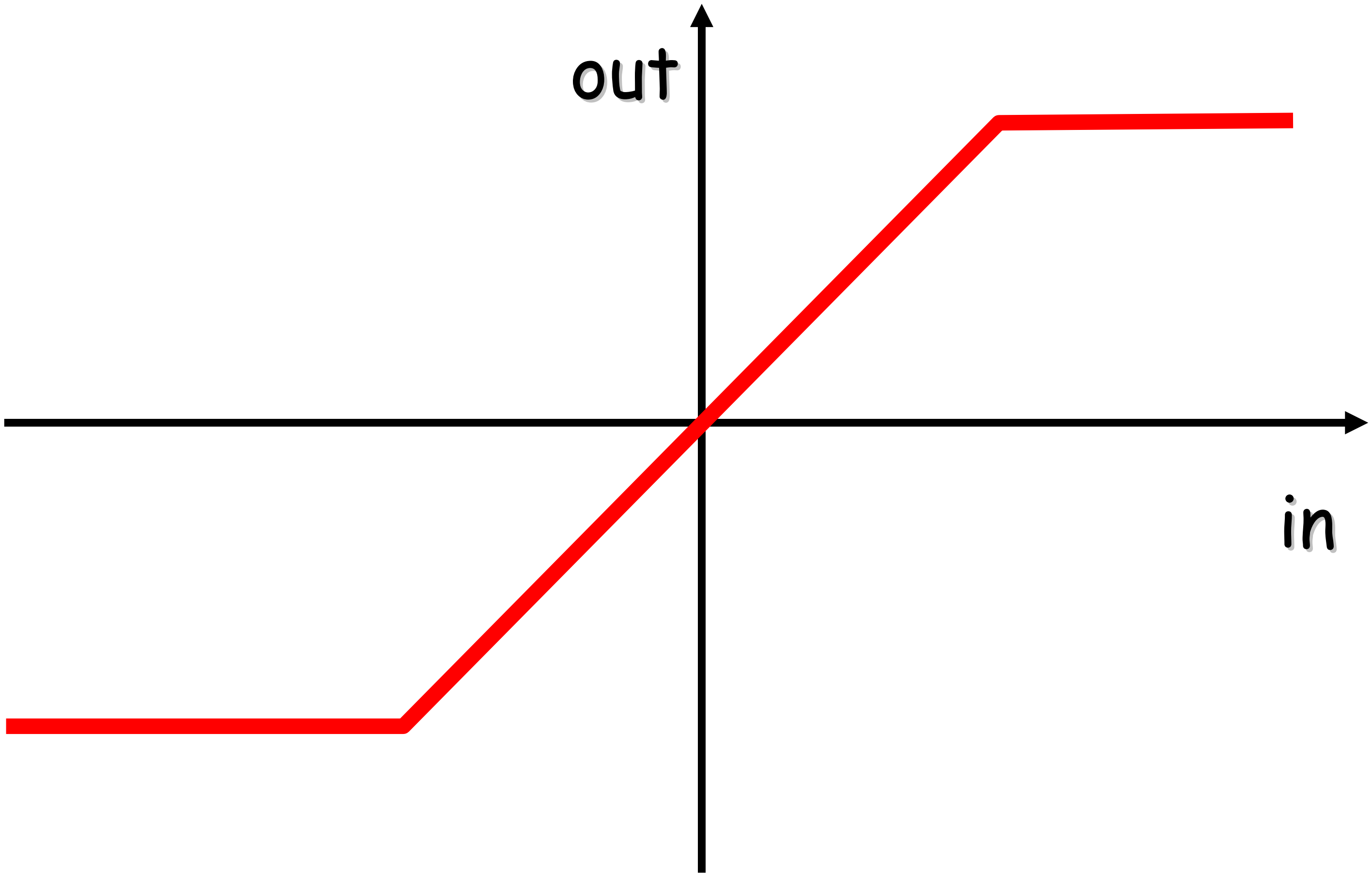
Saturation



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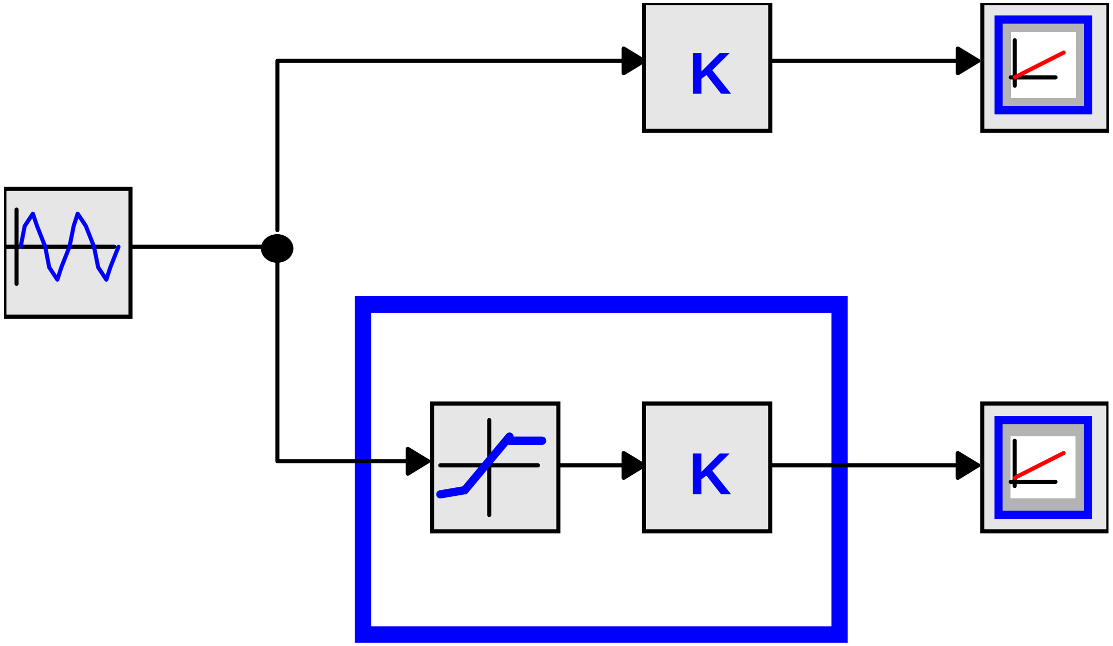
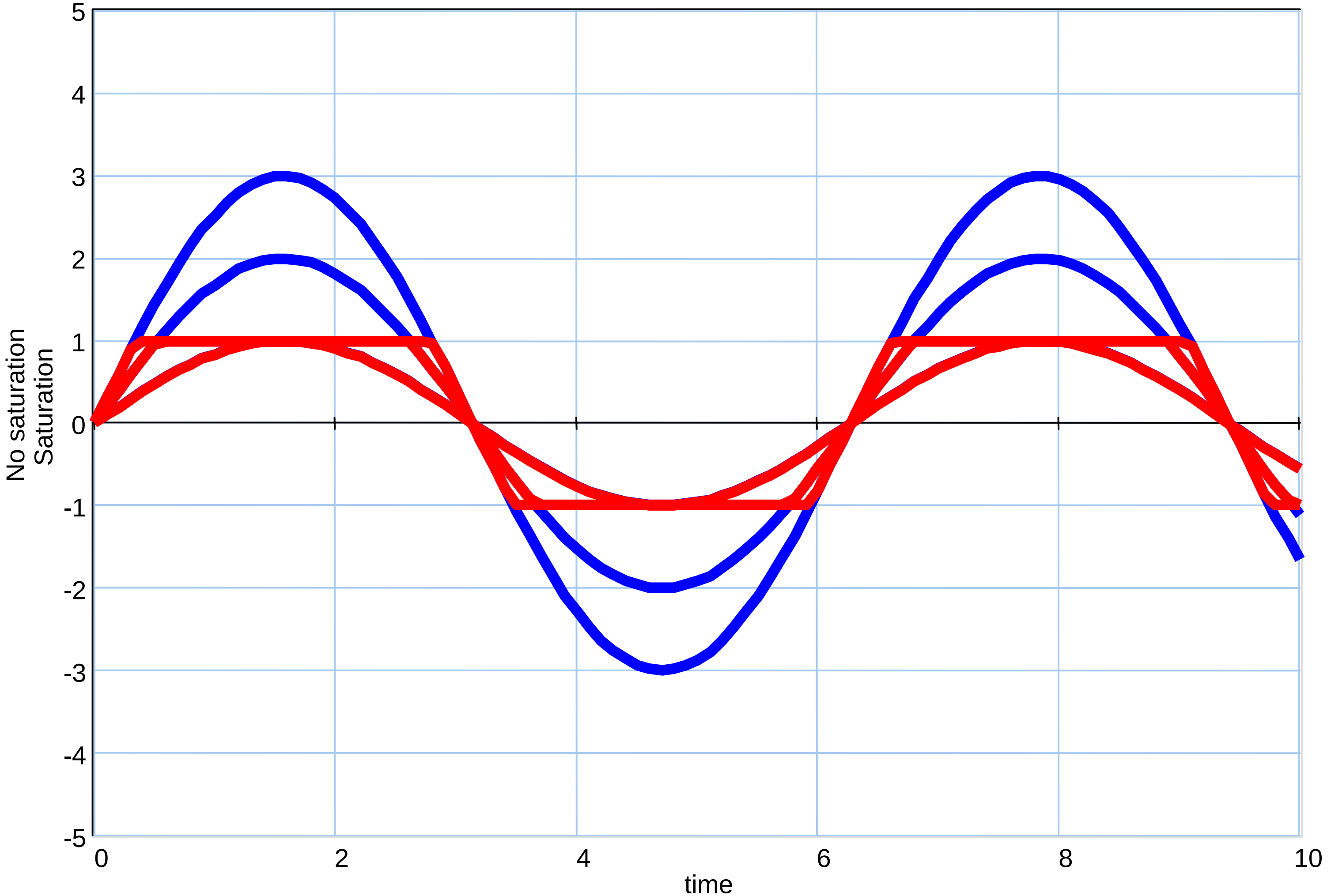
Saturation in amplifier



20-sim: Saturation_demo

Saturation in amplifier



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Thermostat

- Temperature control in
  - buildings
  - boilers
  - refrigerators
  - cars (motor cooling)

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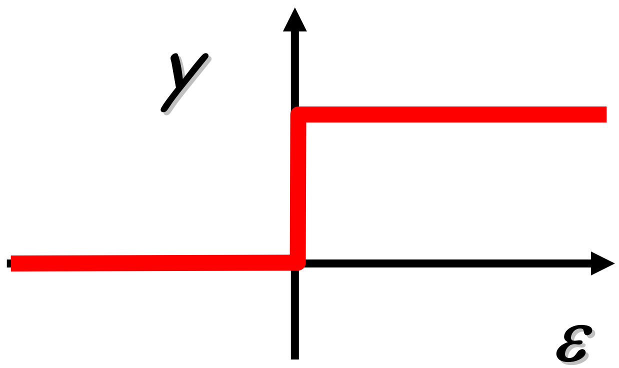
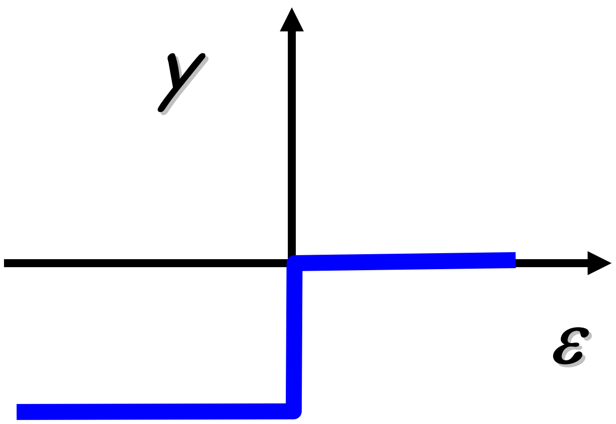
Thermostat





relay:
if $x > 0$, $y = y_{max}$
else $y = 0$

different control systems
for heating and cooling

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Thermostat with dead zone

It makes no sense to
damp the overshoot caused
by the heating systems by
switching on the
air conditioning

In practice only one
of the two systems active

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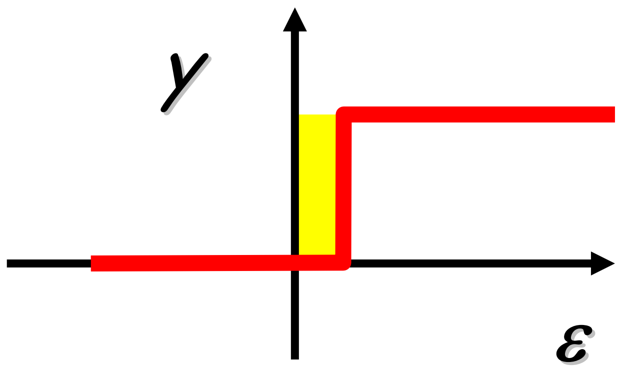
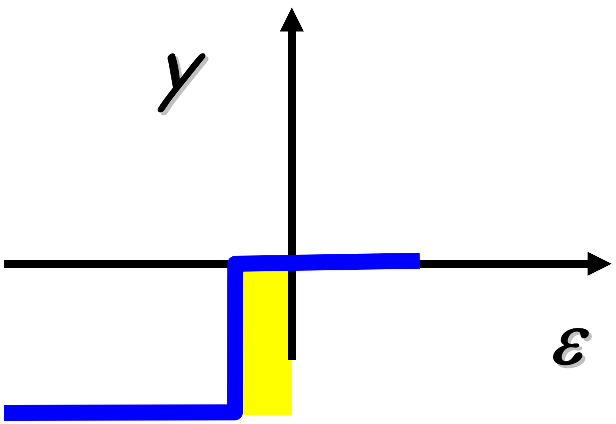
Thermostat with dead zone

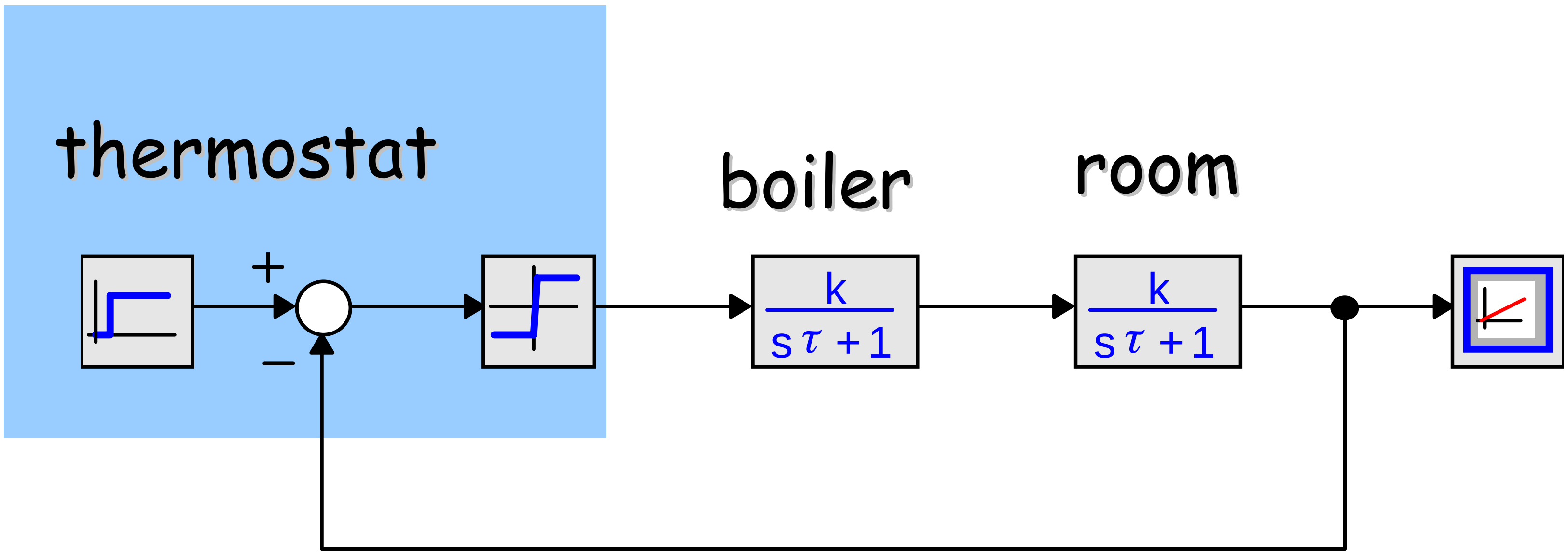




It makes no sense to damp the overshoot caused by the heating systems by switching on the air conditioning

In practice only one of the two systems active

Room control



20-sim: Thermostat_demo1

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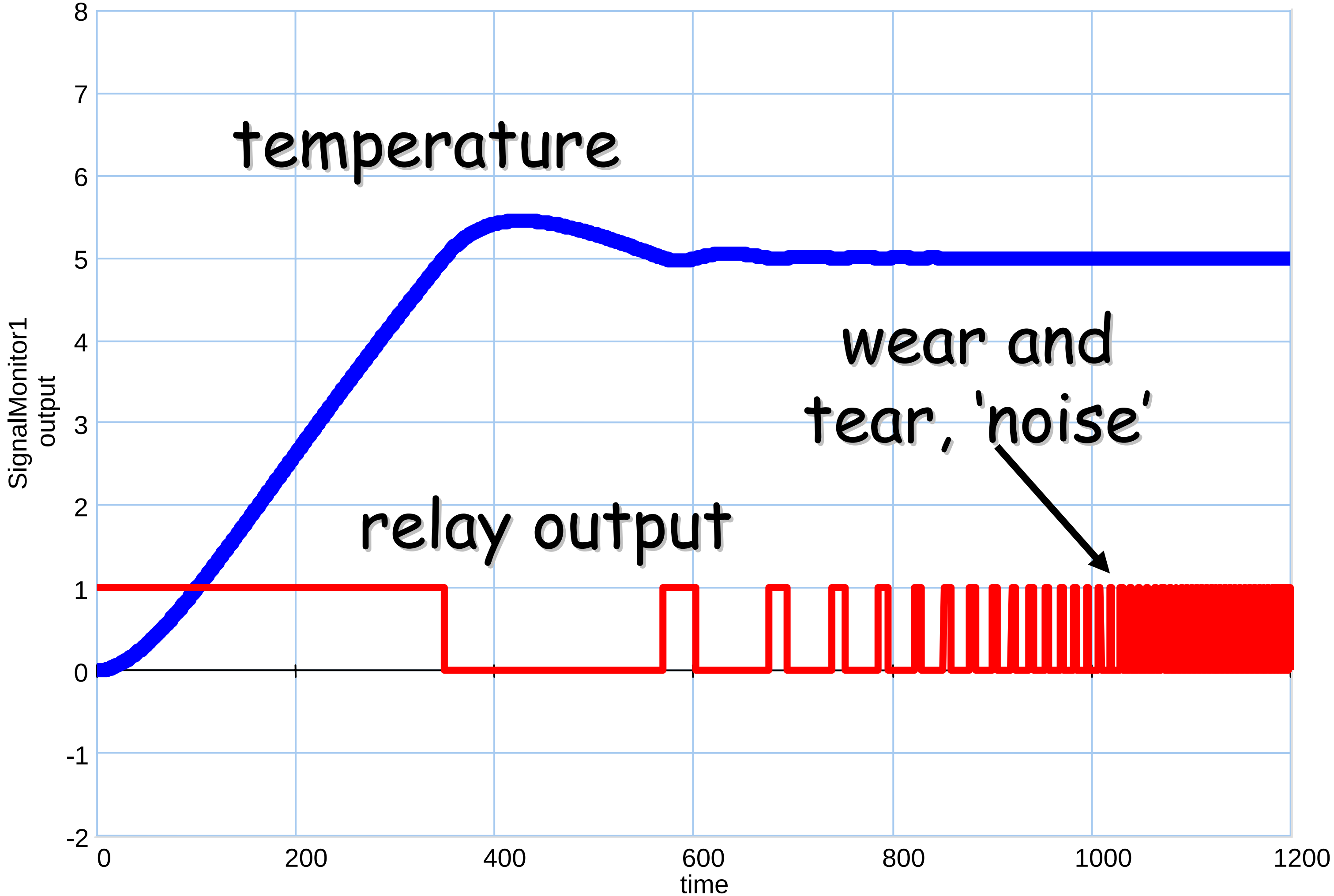
Room control



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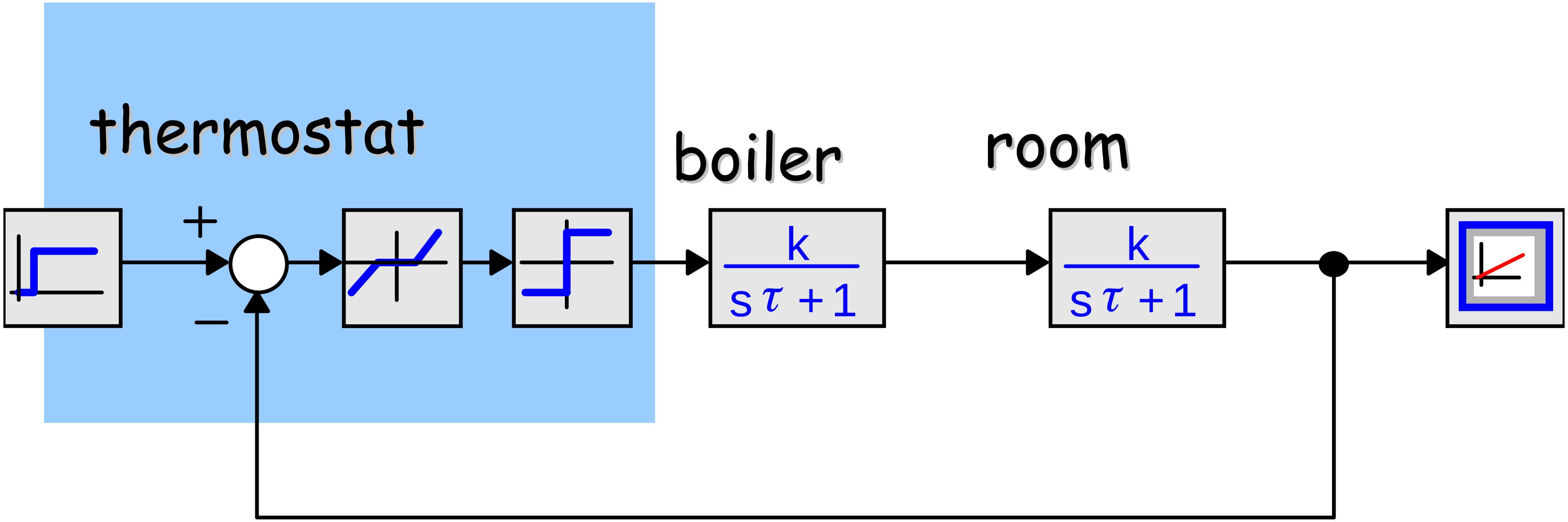
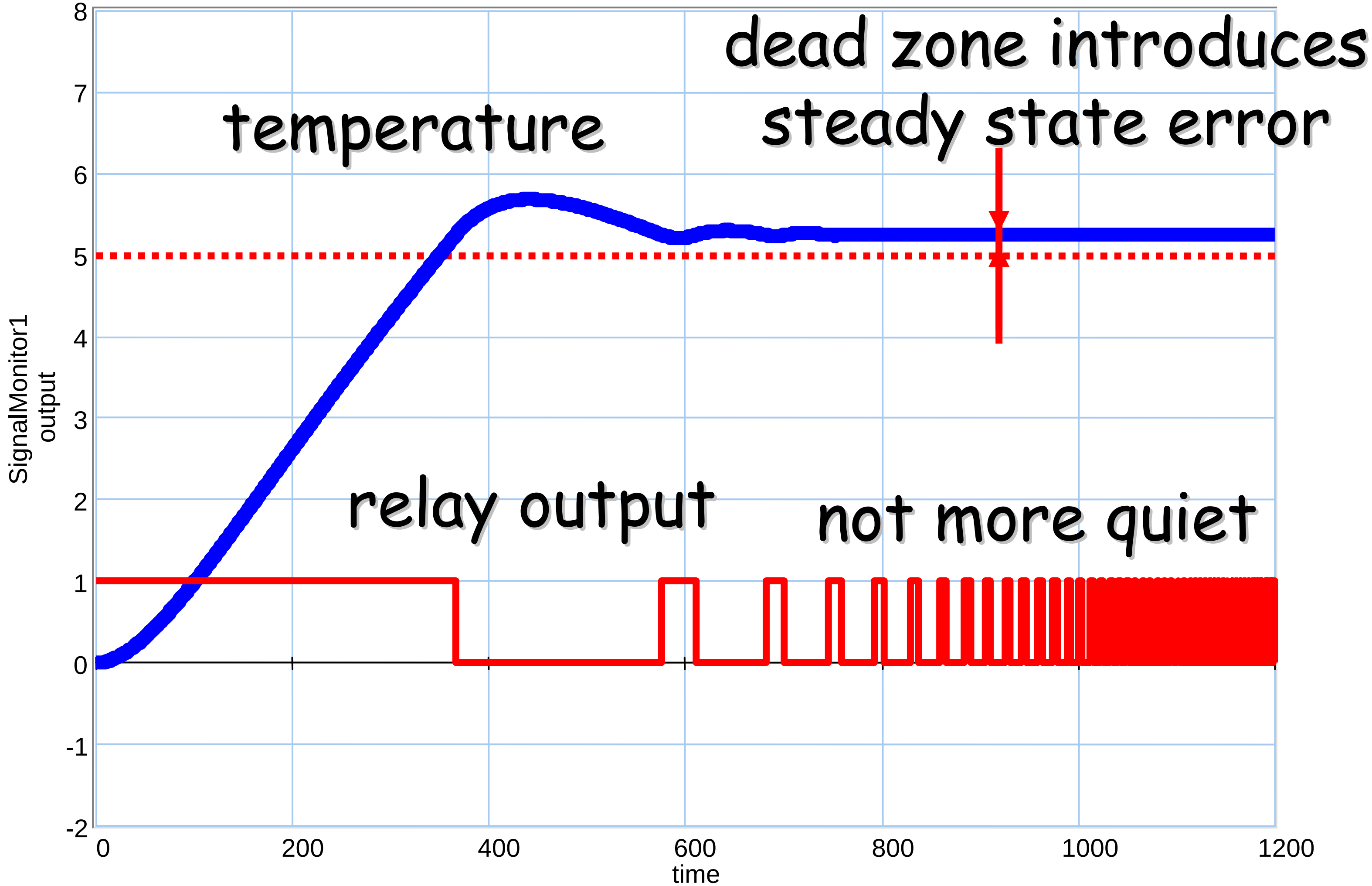
Relay plus dead zone



20-sim: Thermostat_demo2

Relay plus dead zone



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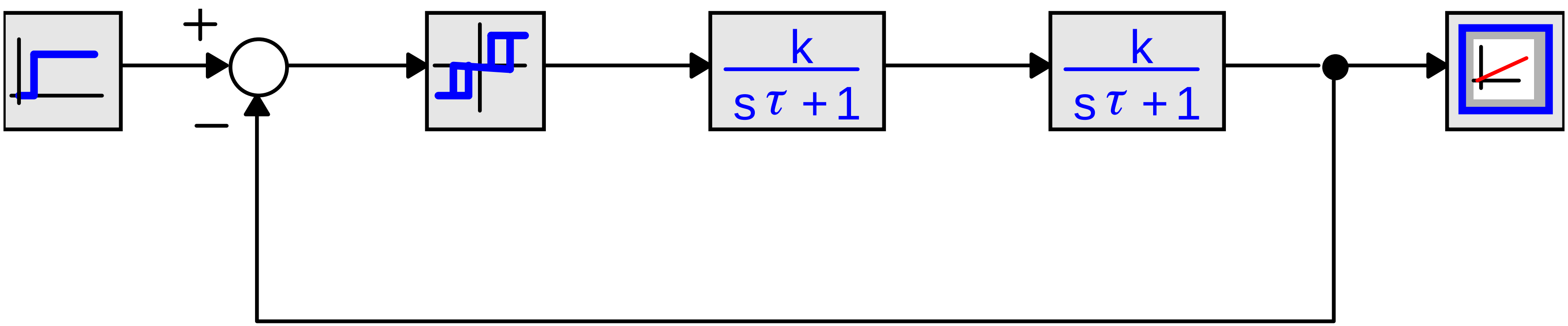
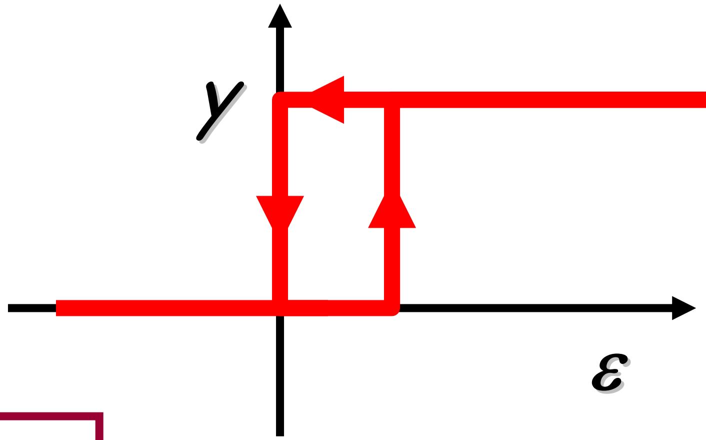
Relay plus hysteresis





20-sim: Thermostat_demo3

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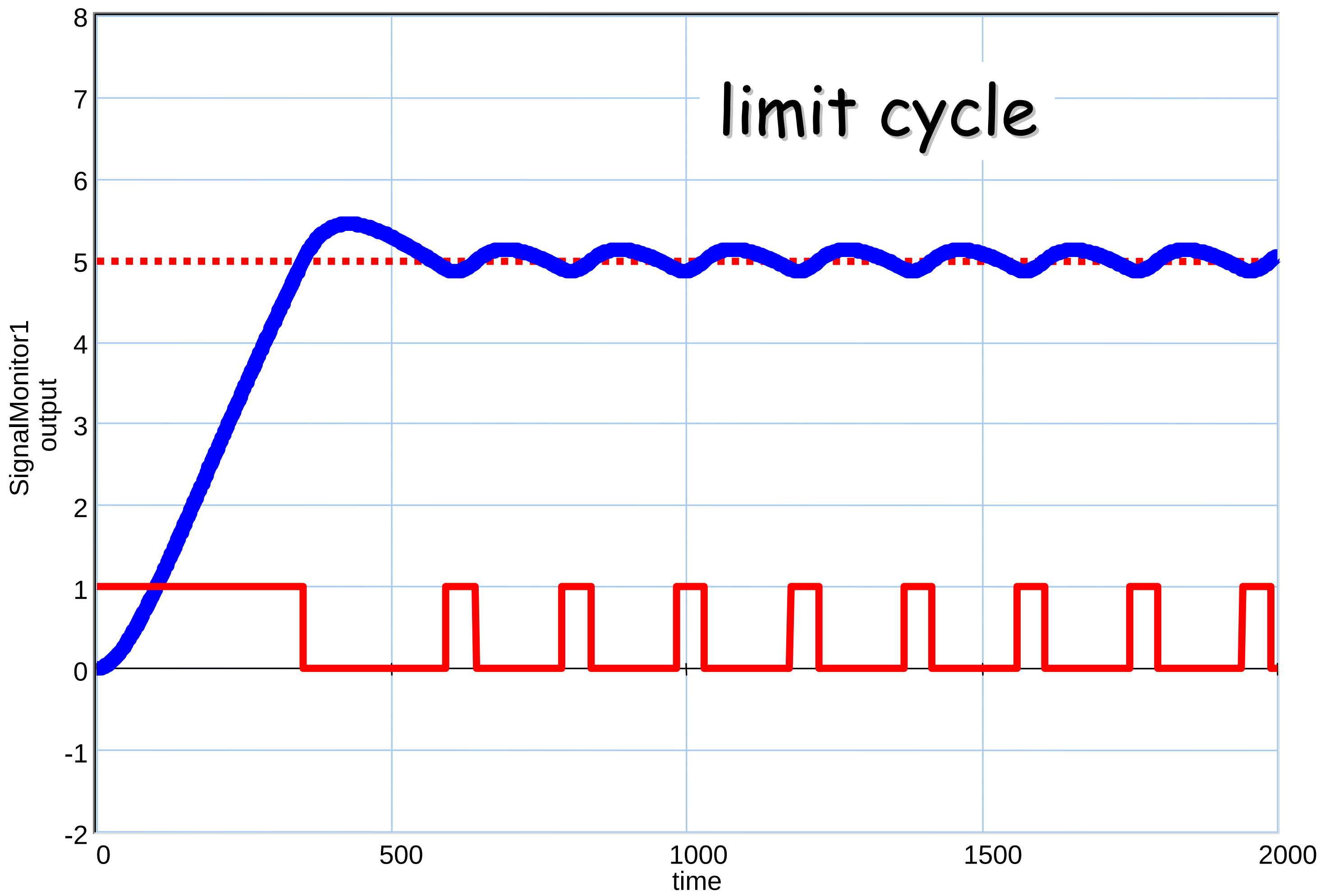
Relay plus hysteresis



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Room thermostat

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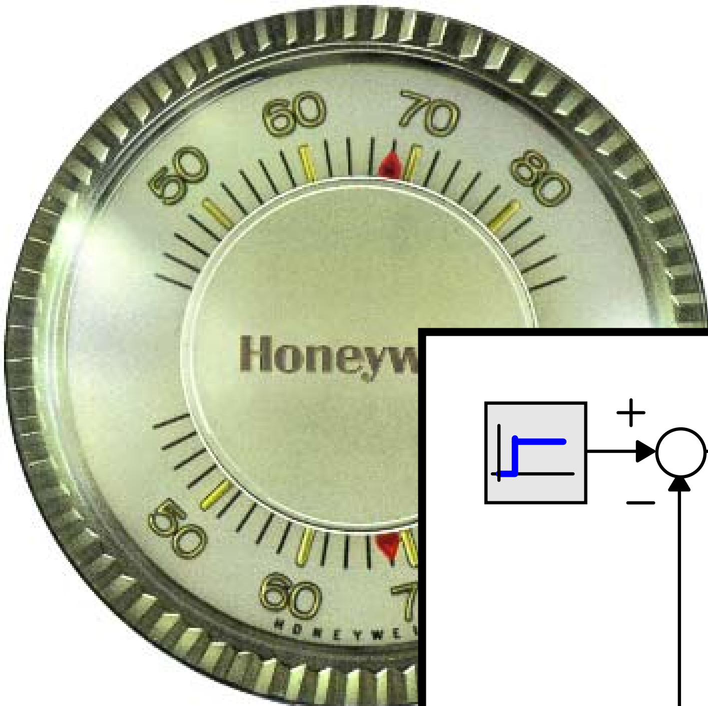
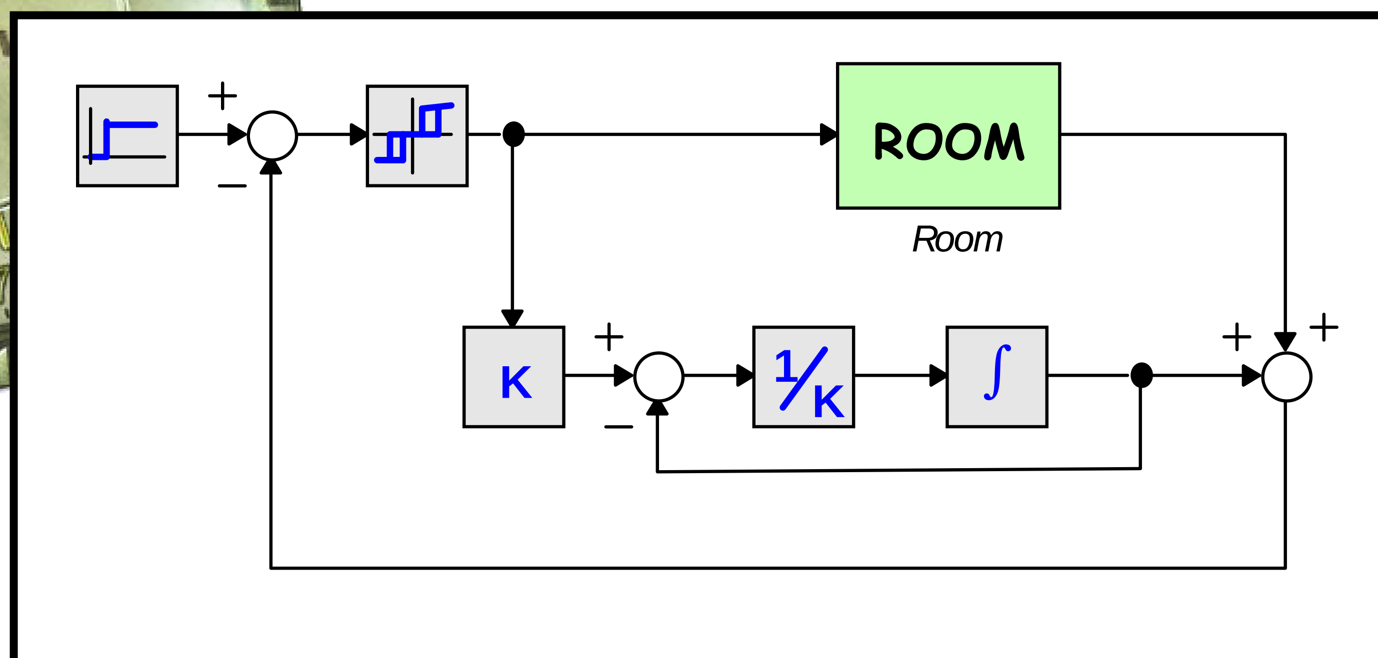


20-sim: CentralHeatingSystem



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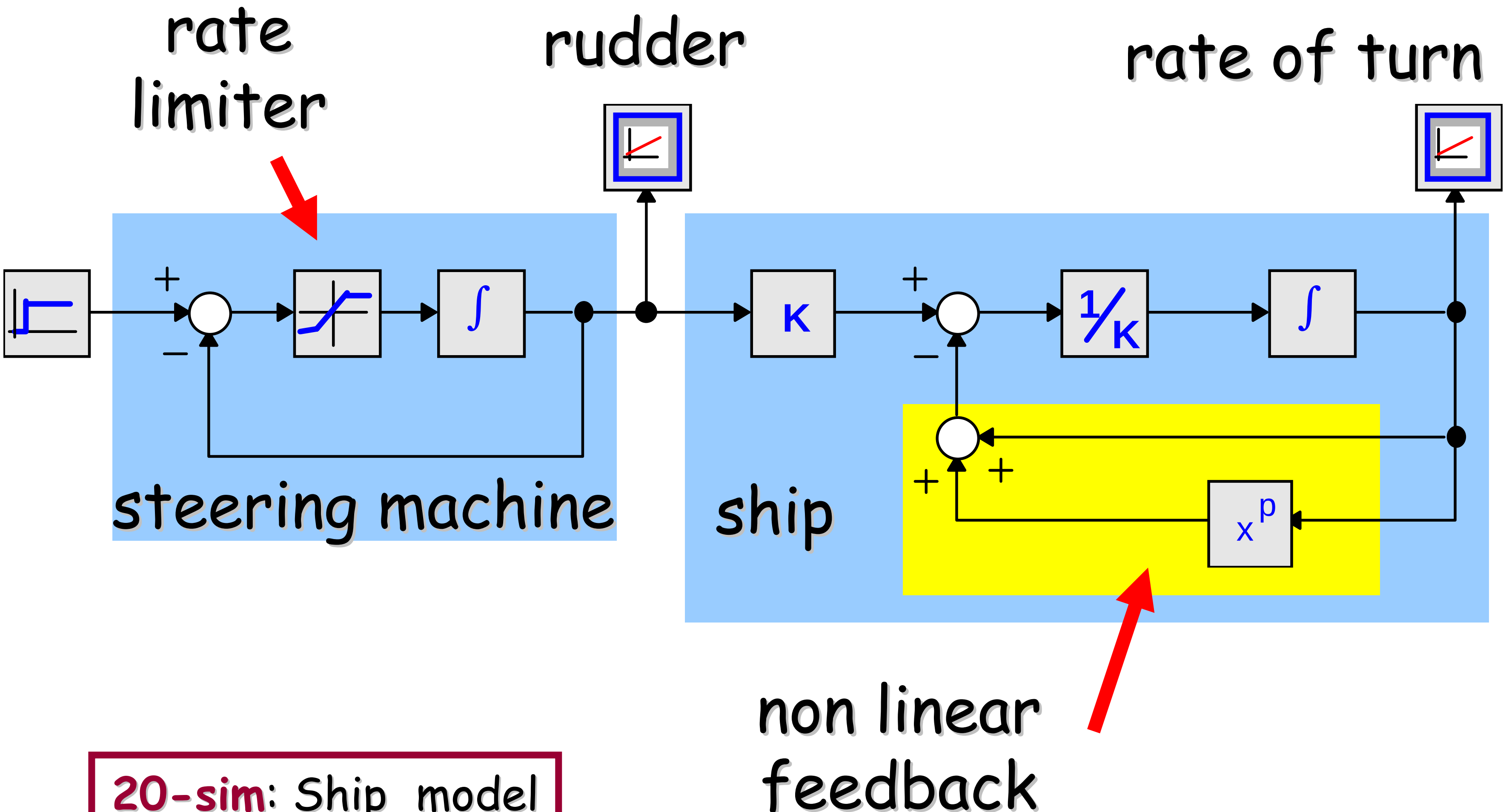
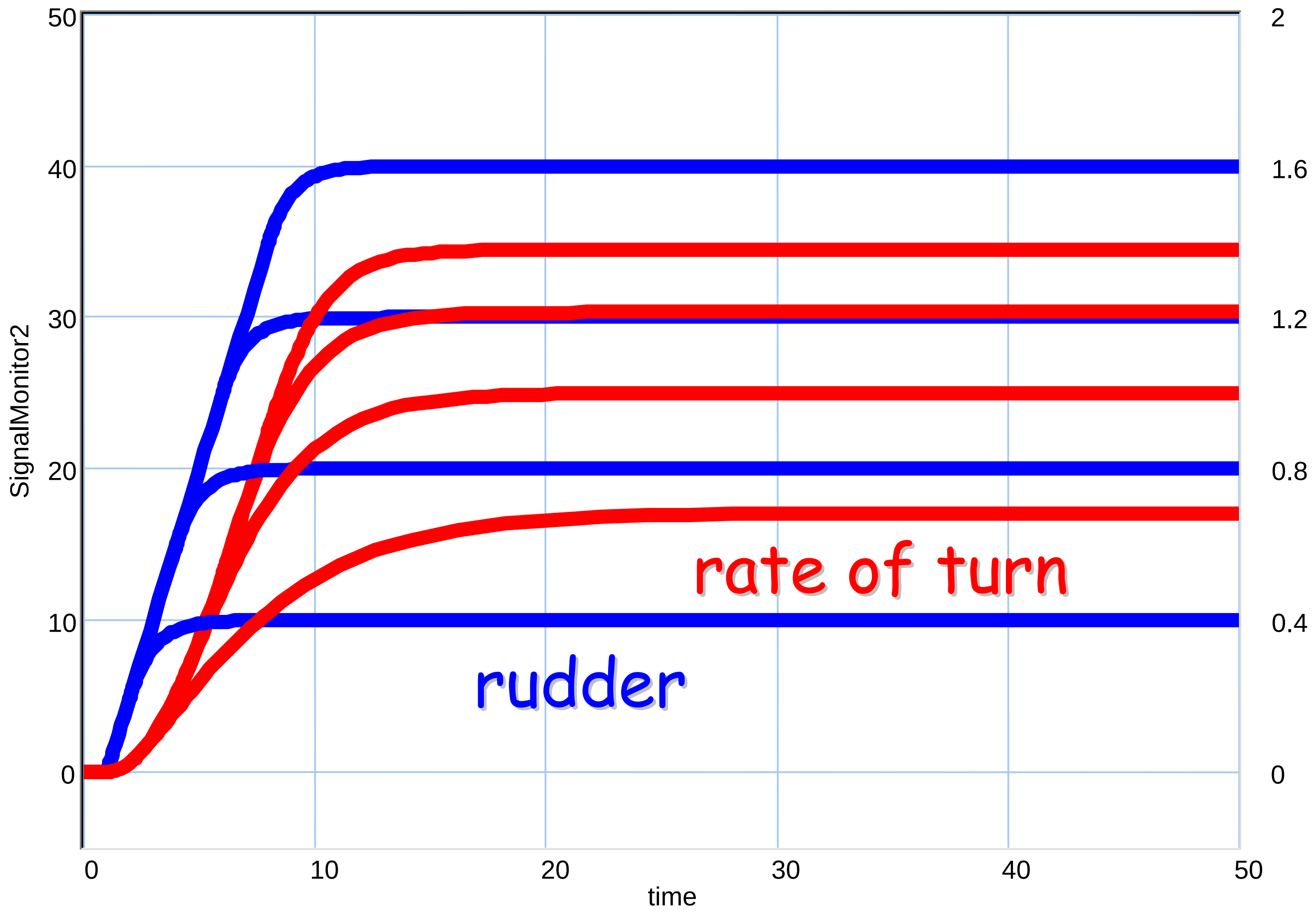
Process properties



20-sim: Ship_model

Process properties



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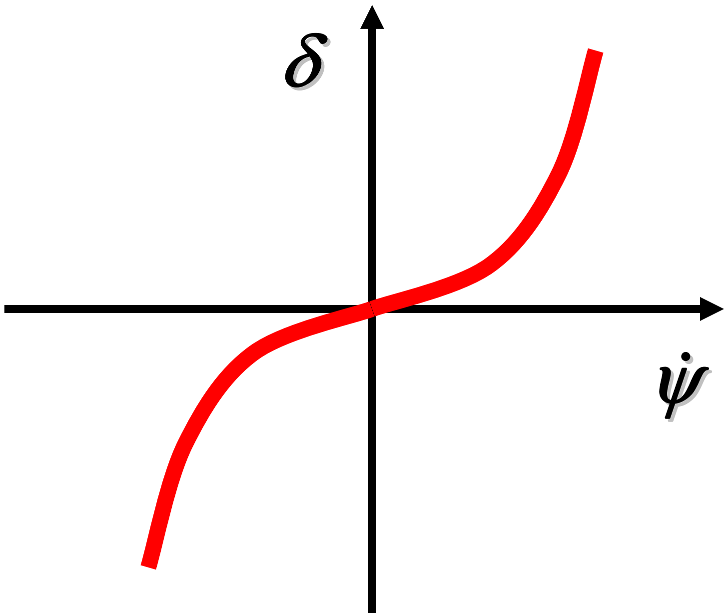
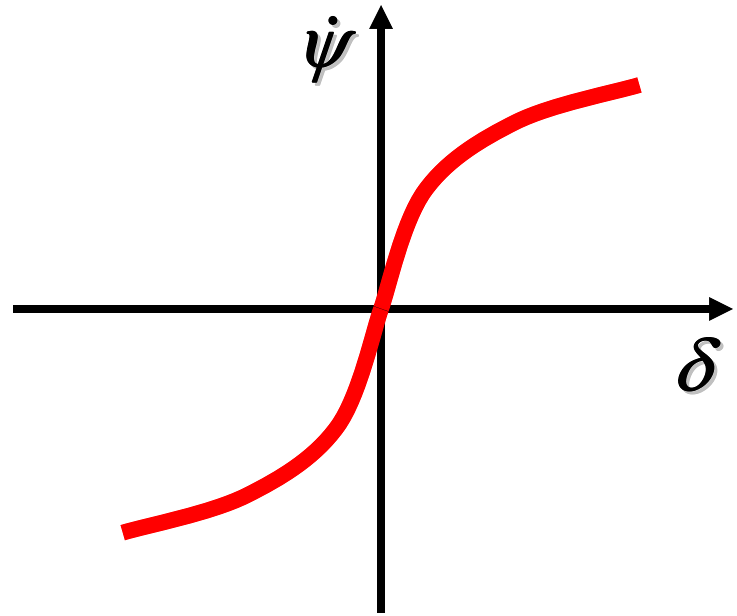
Non linearity



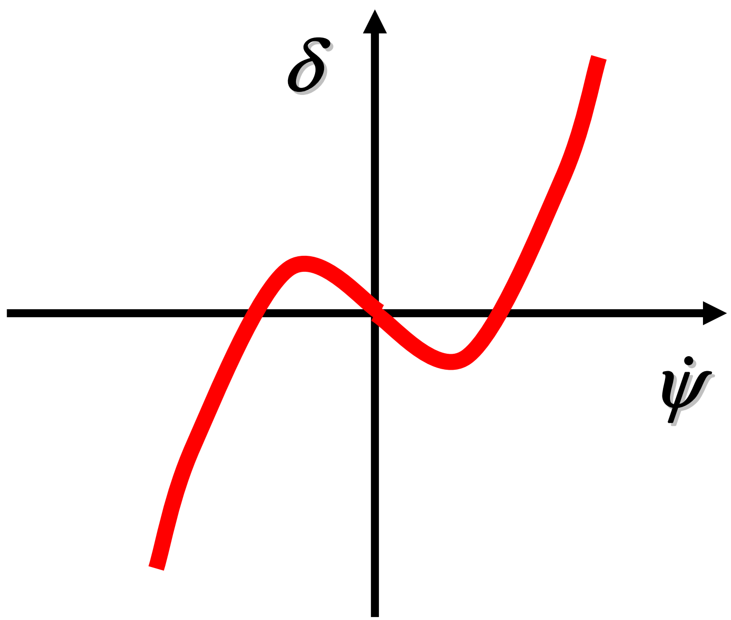


In feedback path:

$$\delta = c_1\dot{\psi} + c_3\dot{\psi}^3$$

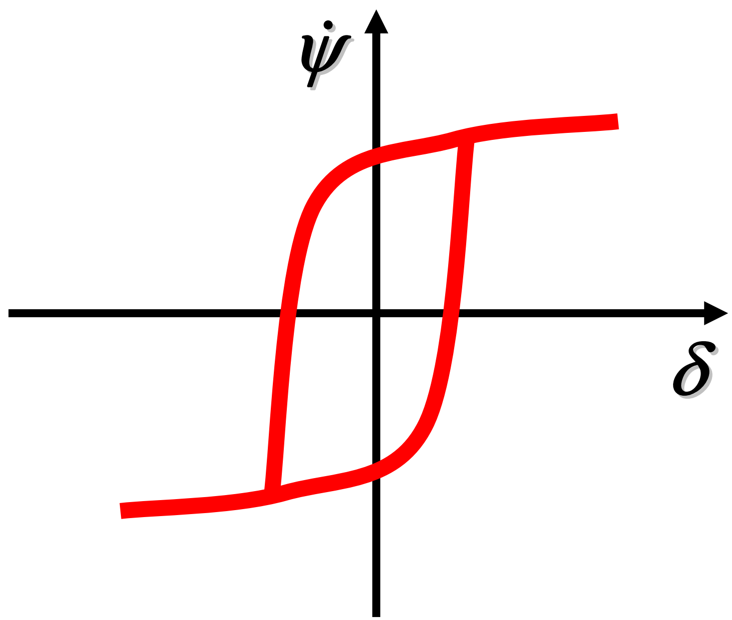
In forward path:

course unstable



In feedback path:

$$\delta = -c_1\dot\psi + c_3\dot\psi^3$$



In forward path:

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Overview

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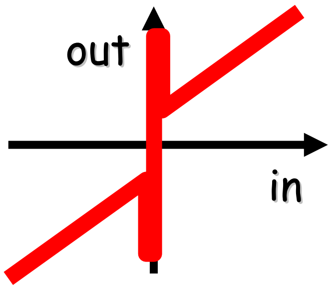
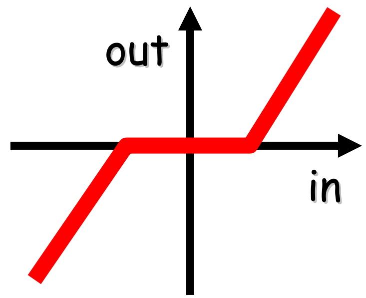
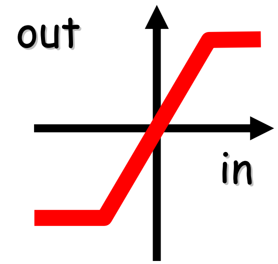


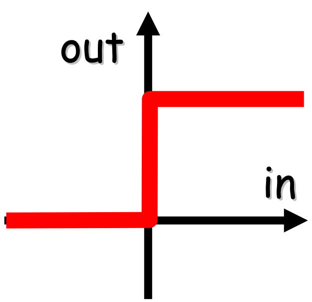
friction



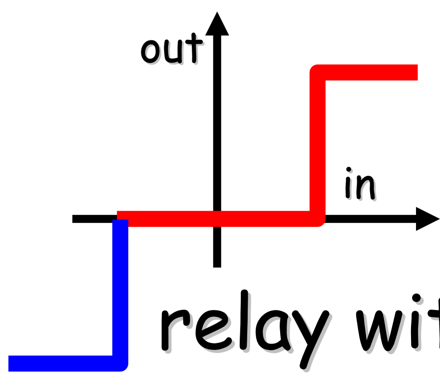
dead zone/band



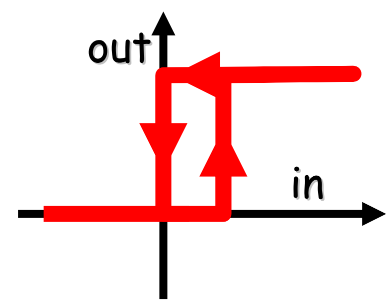
saturation



relay



relay with dead zone



relay with hysteresis

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Why non linearities?

- Sometimes it is difficult or too expensive to make a linear system
  - friction, saturation
- Sometimes wanted
  - safety (maximum relay)
- It is Cheap
  - relay is a cheap power amplifier
    e.g. thermostat, integrated with other functions: sensing, setpoint

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Unwanted non linearities

- Behaviour of a good feedback system is only determined by the elements in the feedback path.

- High gain feedback can compensate for unwanted non-linearities, such as friction

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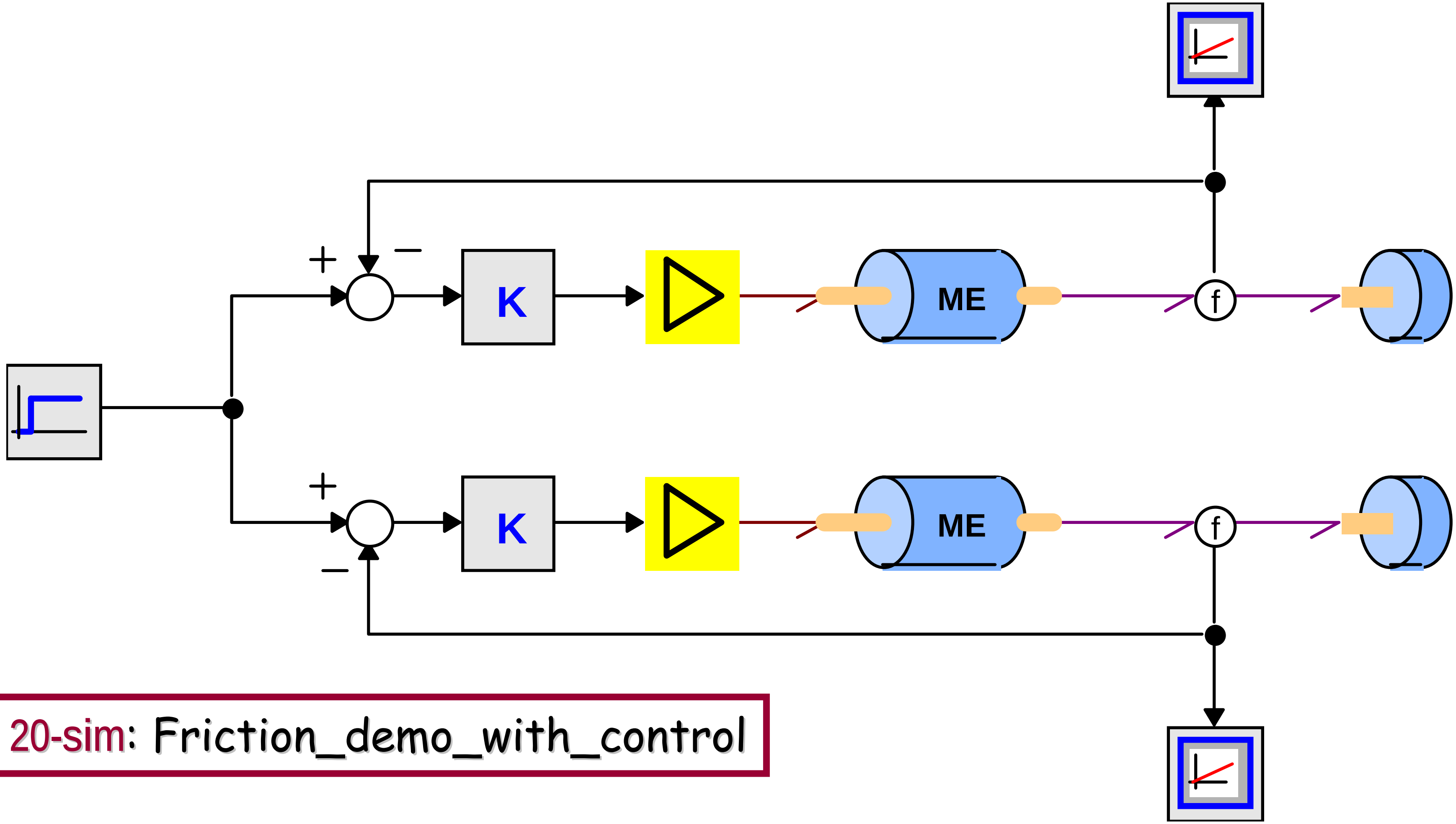
Tacho feedback



20-sim: Friction_demo_with_control

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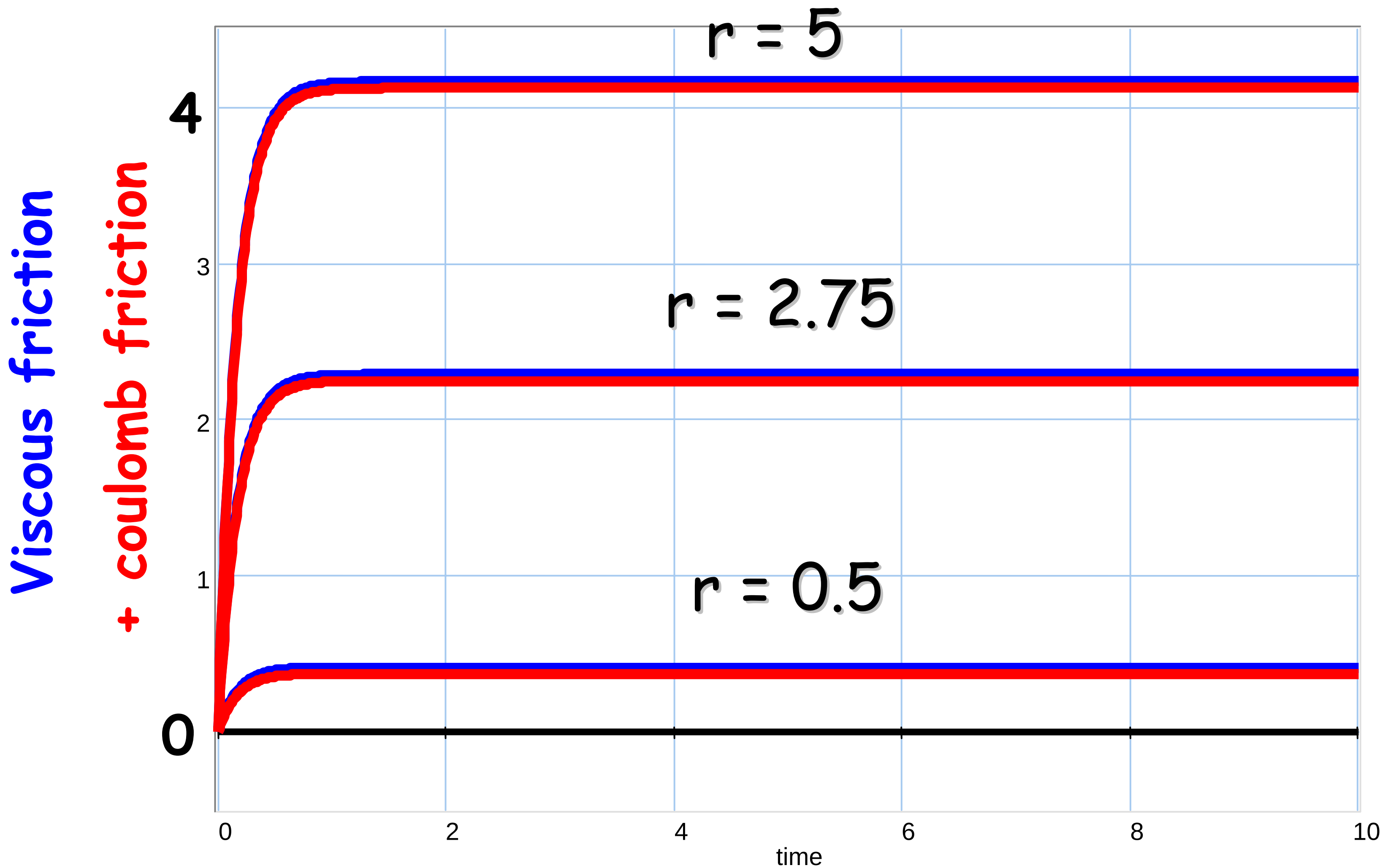
Influence of non linearities with tacho feedback



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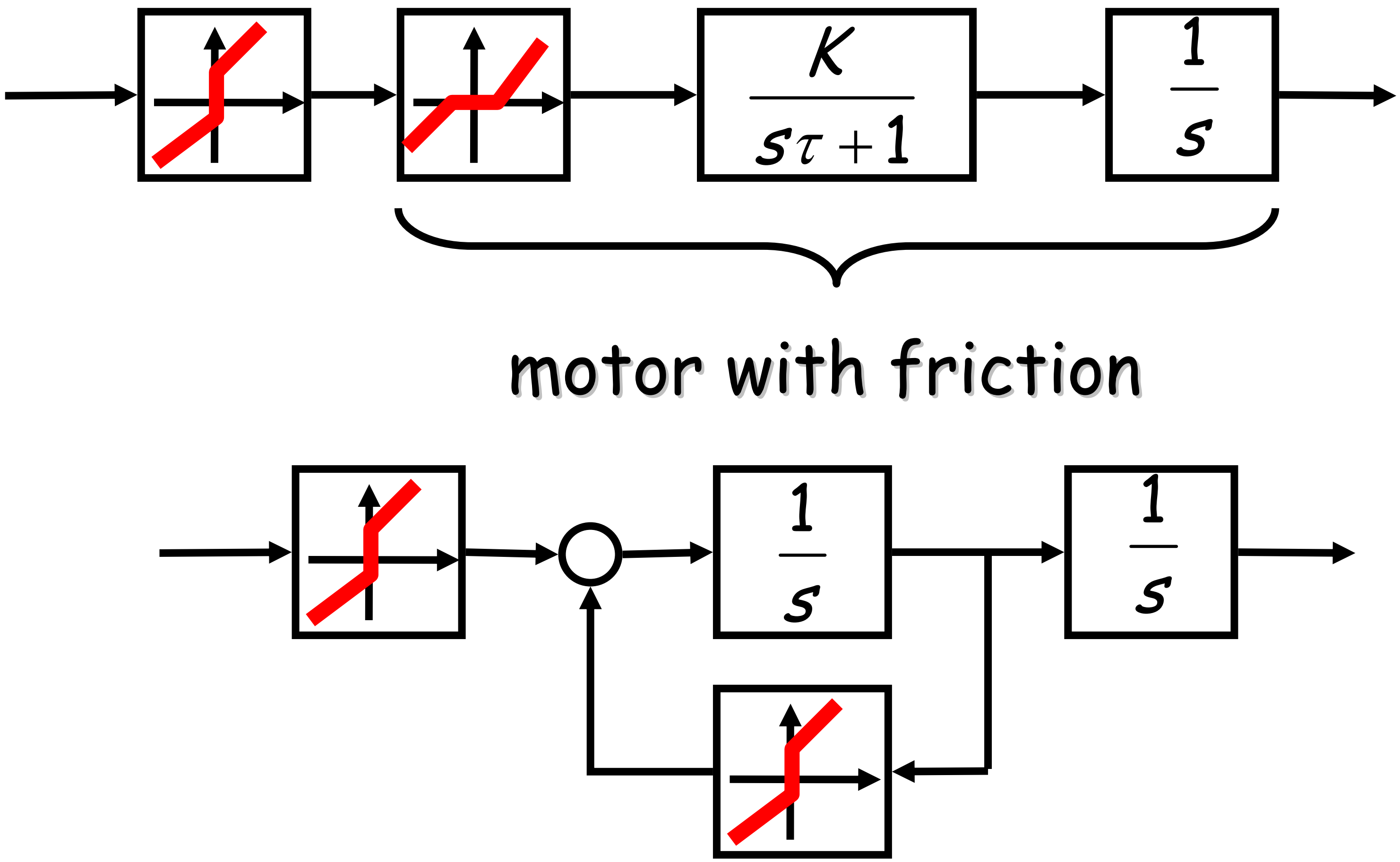
Conclusion

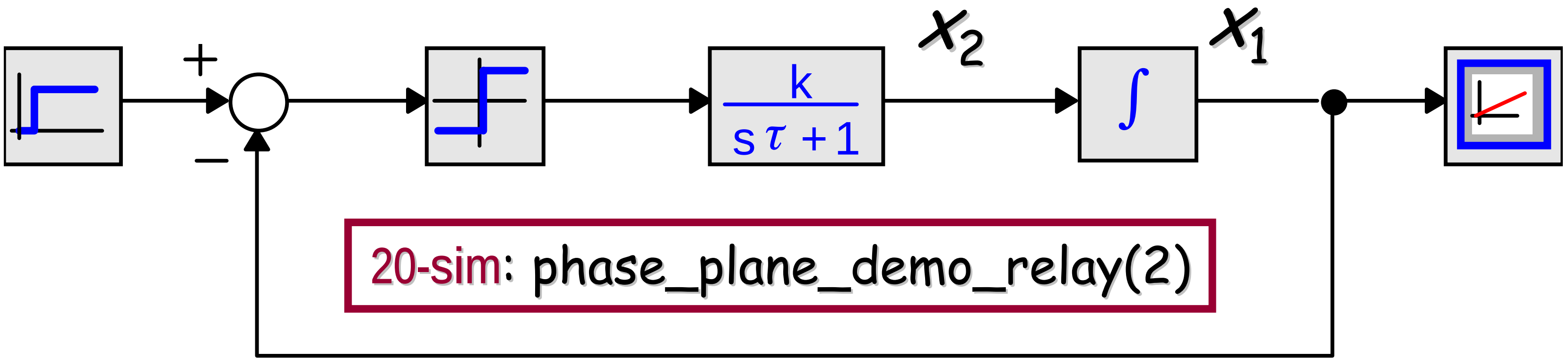
- Feedback can considerably reduce the influence of non linearities
- Especially tacho feedback is an effective means to get rid of the deteriorating effects of static and coulomb friction
- Other possibility:
  inverse characteristic in series
  - much more sensitive for parameters

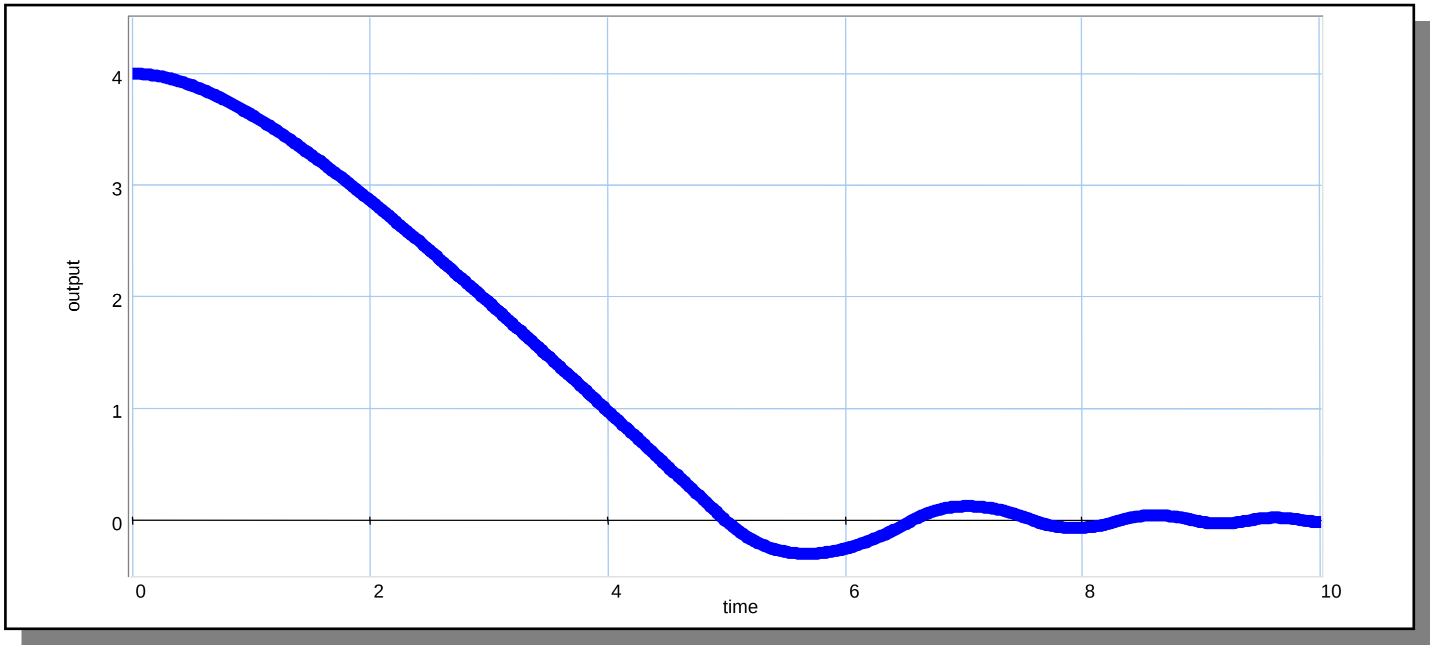
Compensation with inverse



Phase plane analysis

Time response





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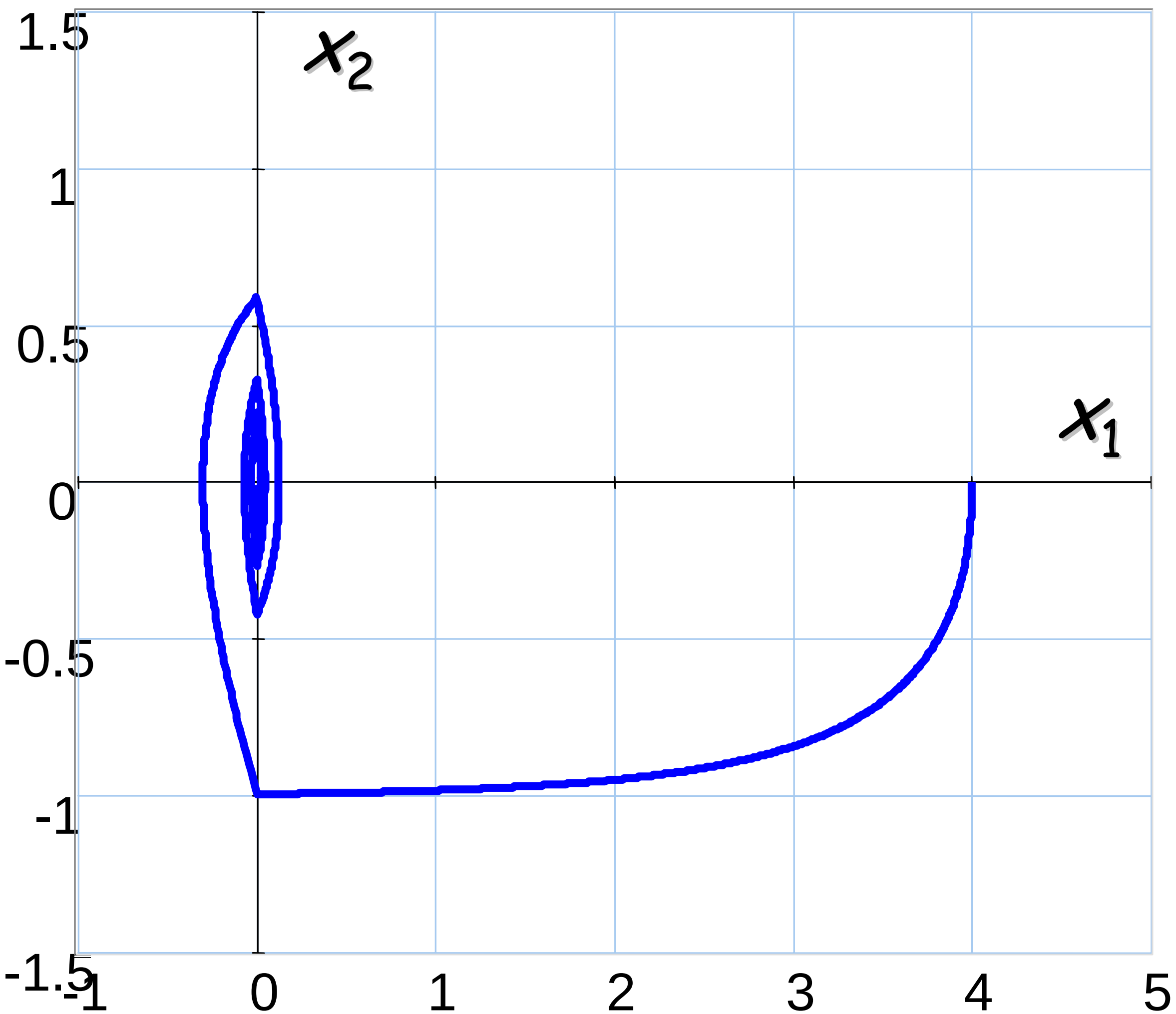
Phase plane plot



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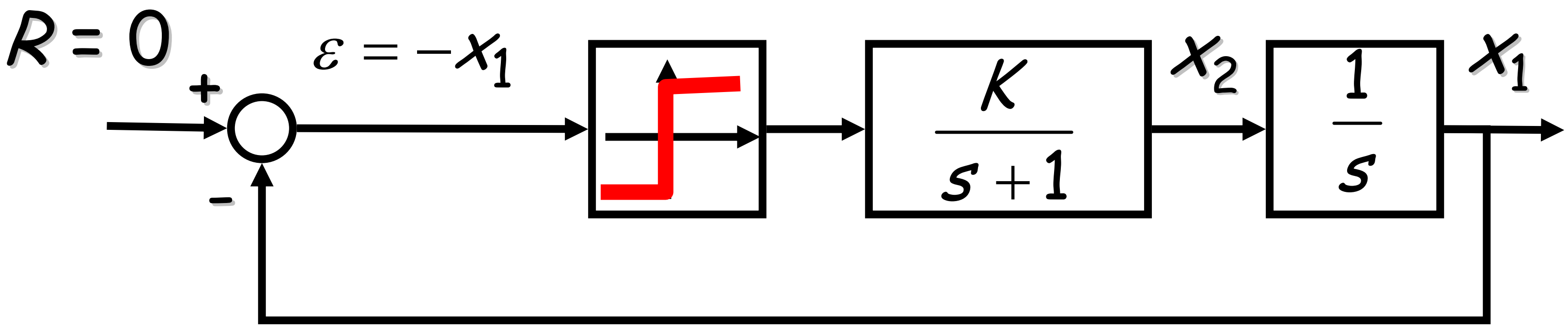
Phase plane analysis



Relay output: $\pm M$

K =1

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 + M \quad \text{if } \varepsilon > 0 \quad (x_1 < 0)$$

$$\dot{x}_2 = -x_2 - M \quad \text{if } \varepsilon < 0 \quad (x_1 > 0)$$

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Isocline

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 + M \quad \text{if } \varepsilon > 0 \quad (x_1 < 0)$$

$$\dot{x}_2 = -x_2 - M \quad \text{if } \varepsilon < 0 \quad (x_1 > 0)$$

isocline

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{-x_2 + M}{x_2} \quad (x_1 < 0)$$

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{-x_2 - M}{x_2} \quad (x_1 > 0)$$

$$\boxed{\begin{aligned} \frac{dx_2}{dx_1} &= -1 + \frac{M}{x_2} = m \\ \frac{dx_2}{dx_1} &= -1 - \frac{M}{x_2} = m \end{aligned}}$$

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Some values

$$m = -1 + \frac{M}{x_2} \quad (x_1 < 0)$$

$$m = -1 - \frac{M}{x_2} \quad (x_1 > 0)$$

m

x_2	$x_1 < 0$	$x_1 > 0$
0	∞	$-\infty$
M	0	-2
$-M$	-2	0
$M/2$	1	-3
$-M/2$	-3	1

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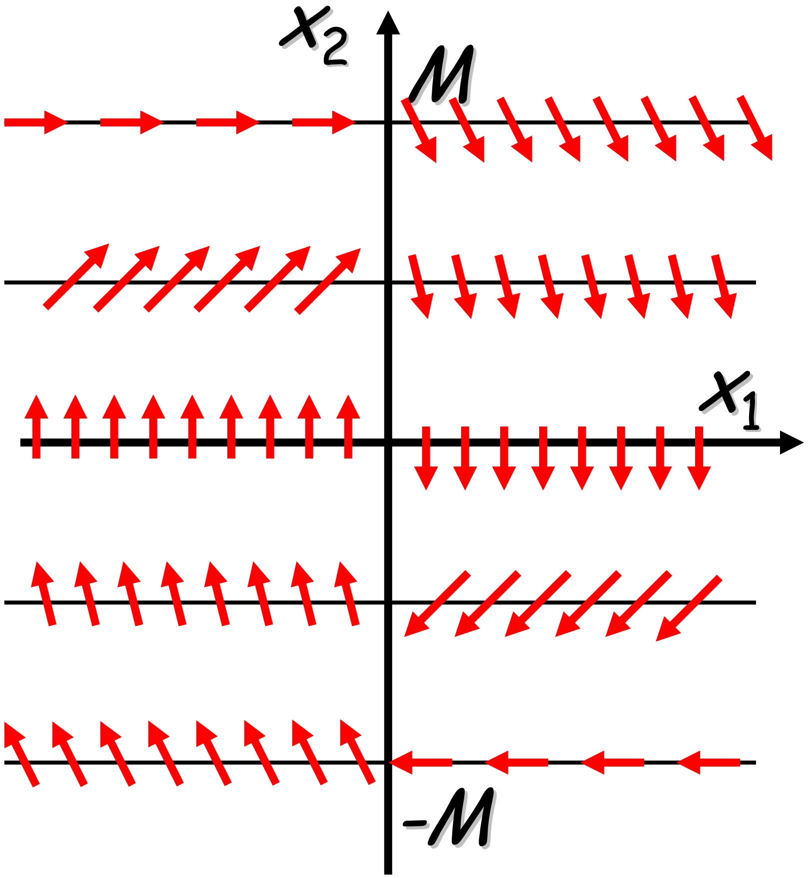
Graphical



x_2	$x_1 < 0$	$x_1 > 0$
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M	0	-2
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$M/2$	1	-3
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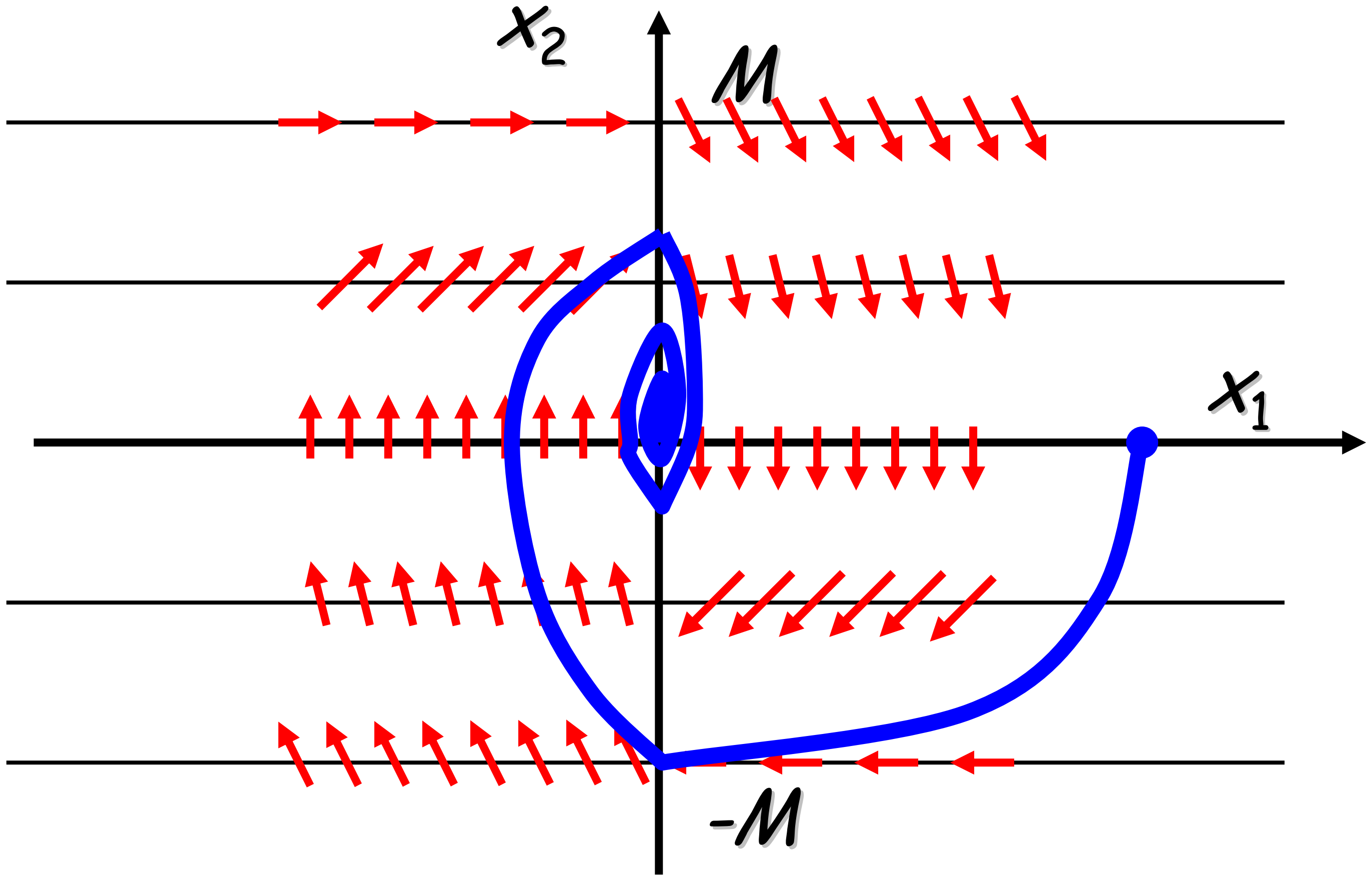
Graphical



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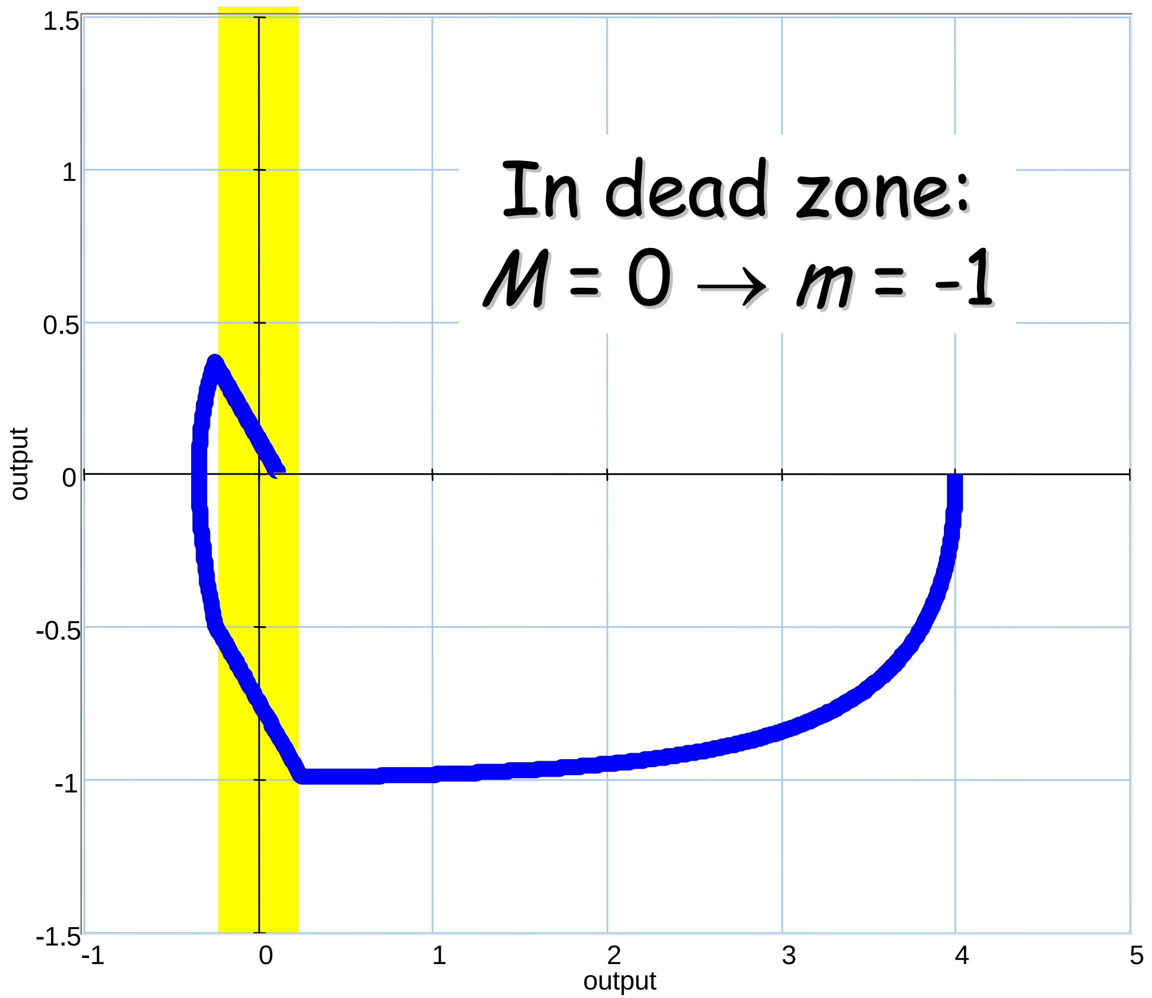
Relay with dead zone



20-sim: Phase_plane_demo_relay_DZ

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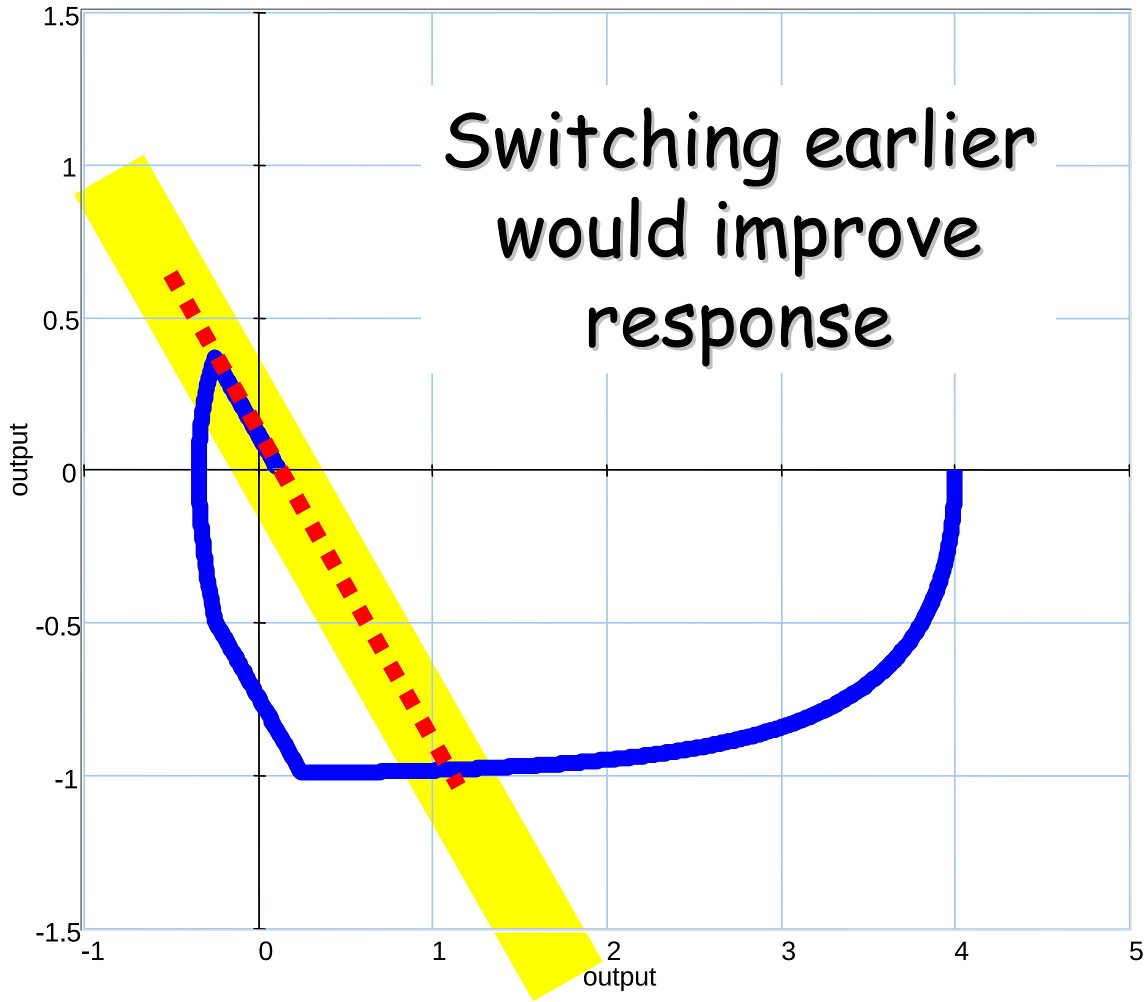
Switching line



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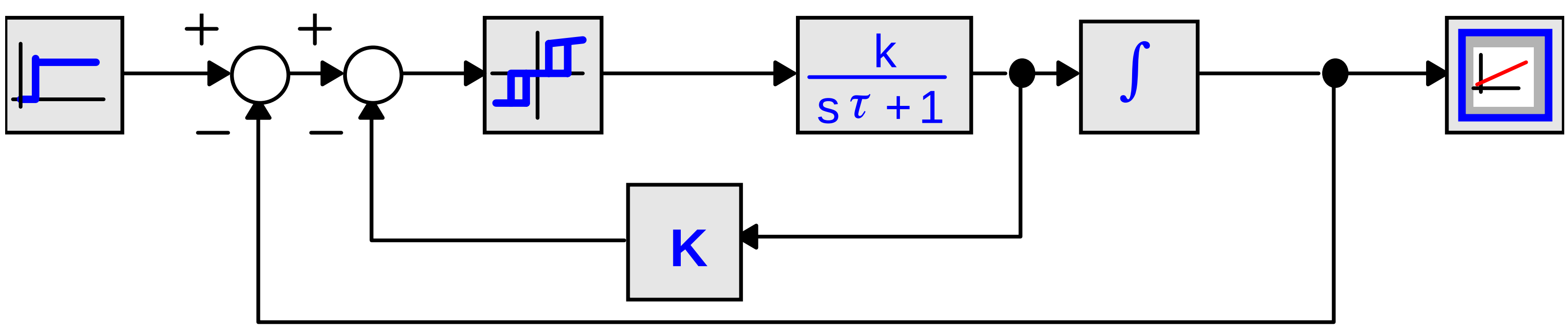
demo



Phaseplane_demo_relay_DZ_switchline

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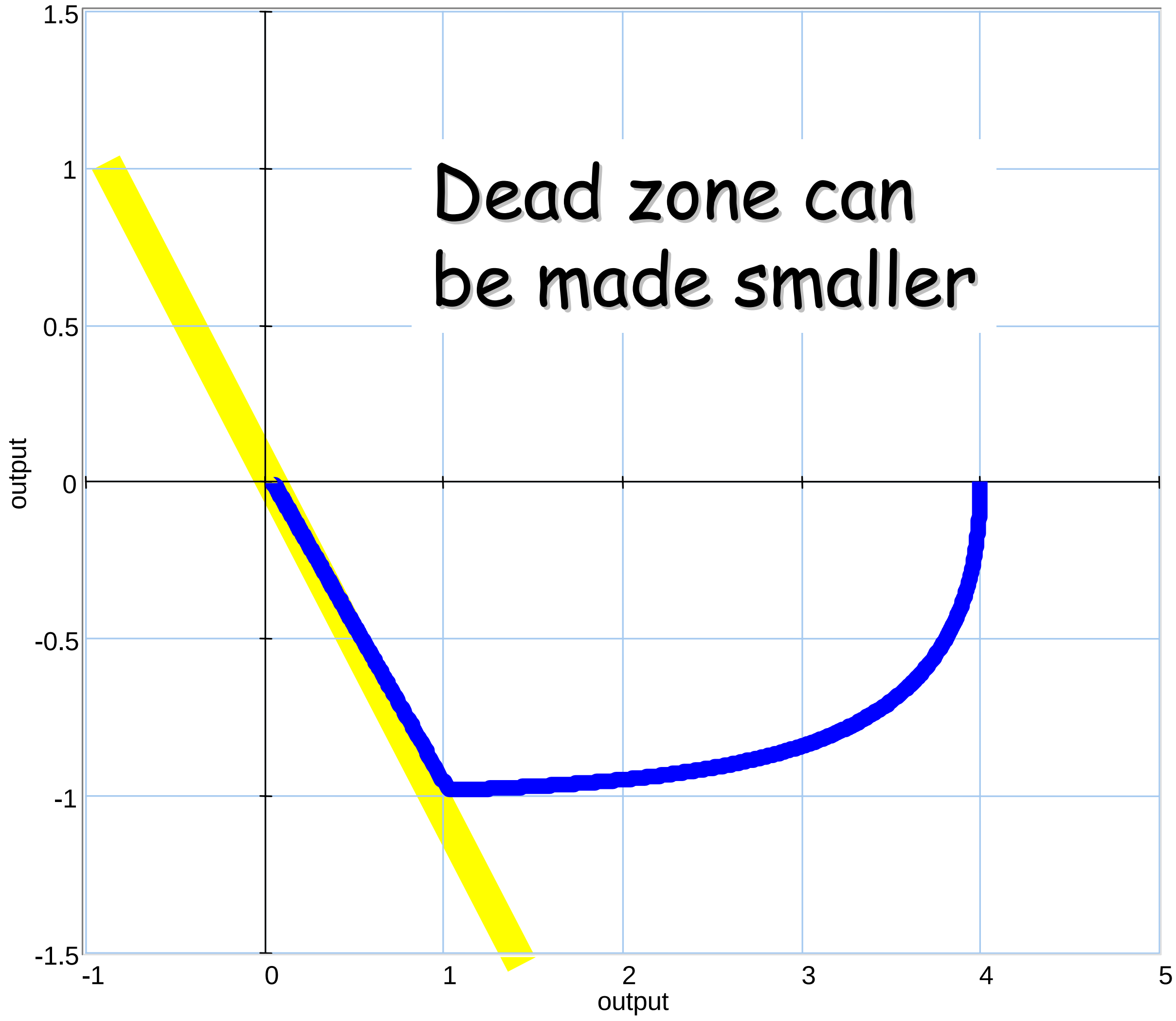
Switching line



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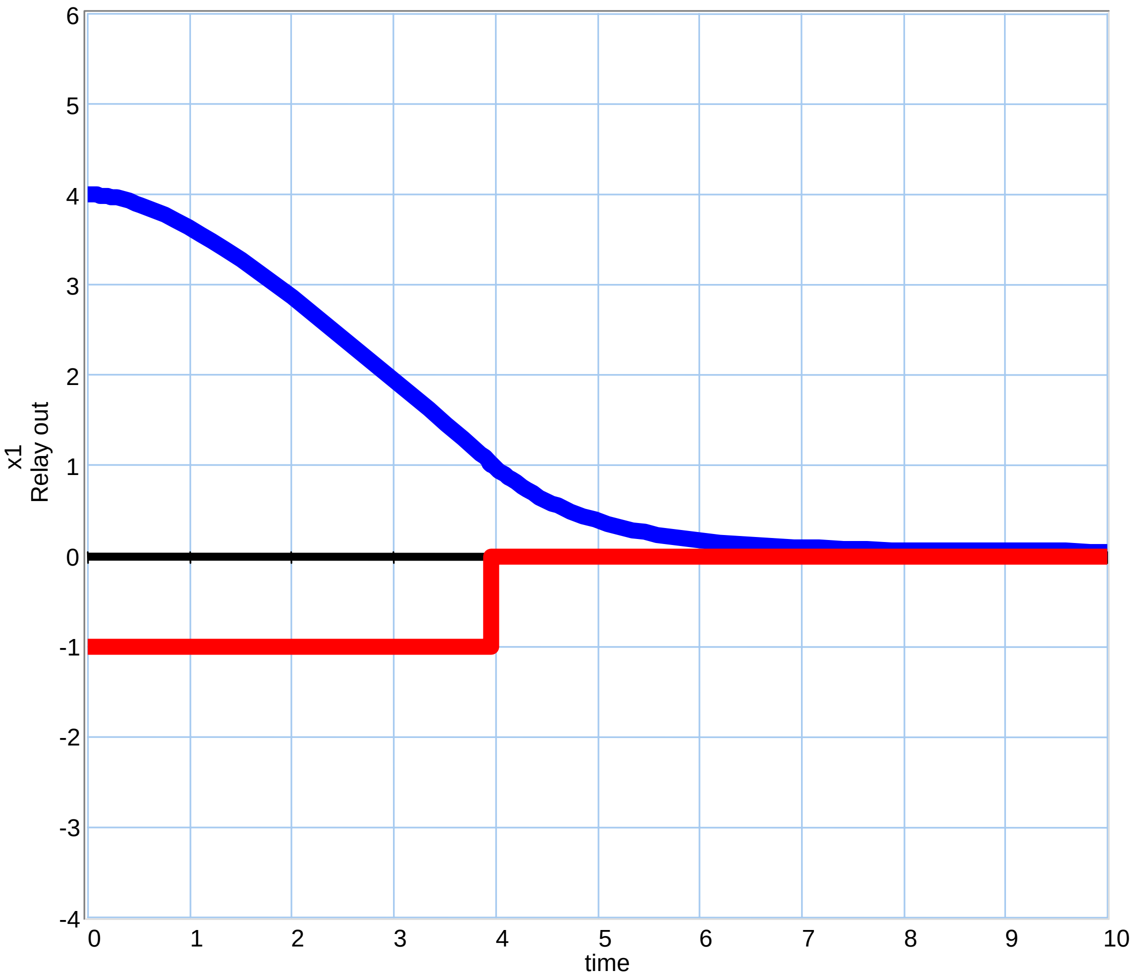
Time response



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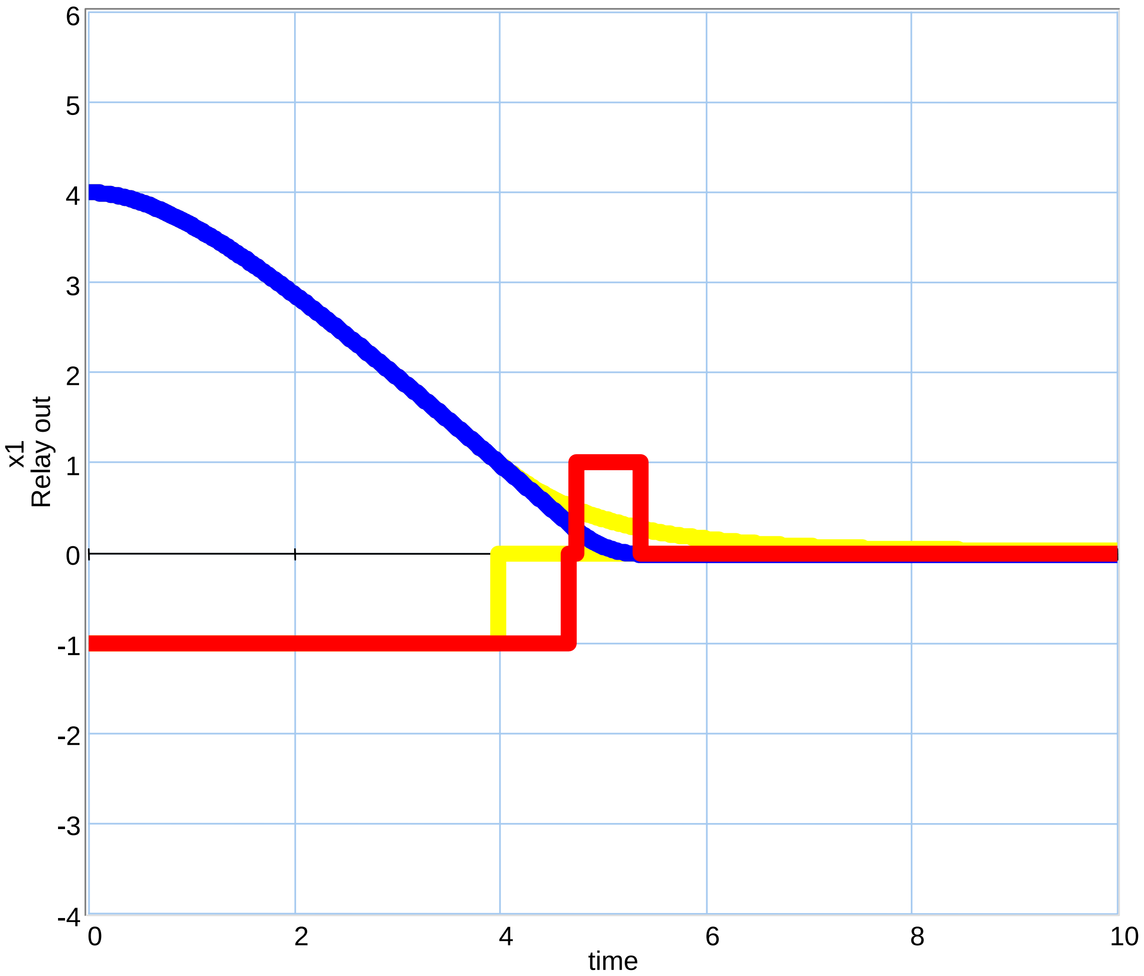
Time optimal (bang bang)



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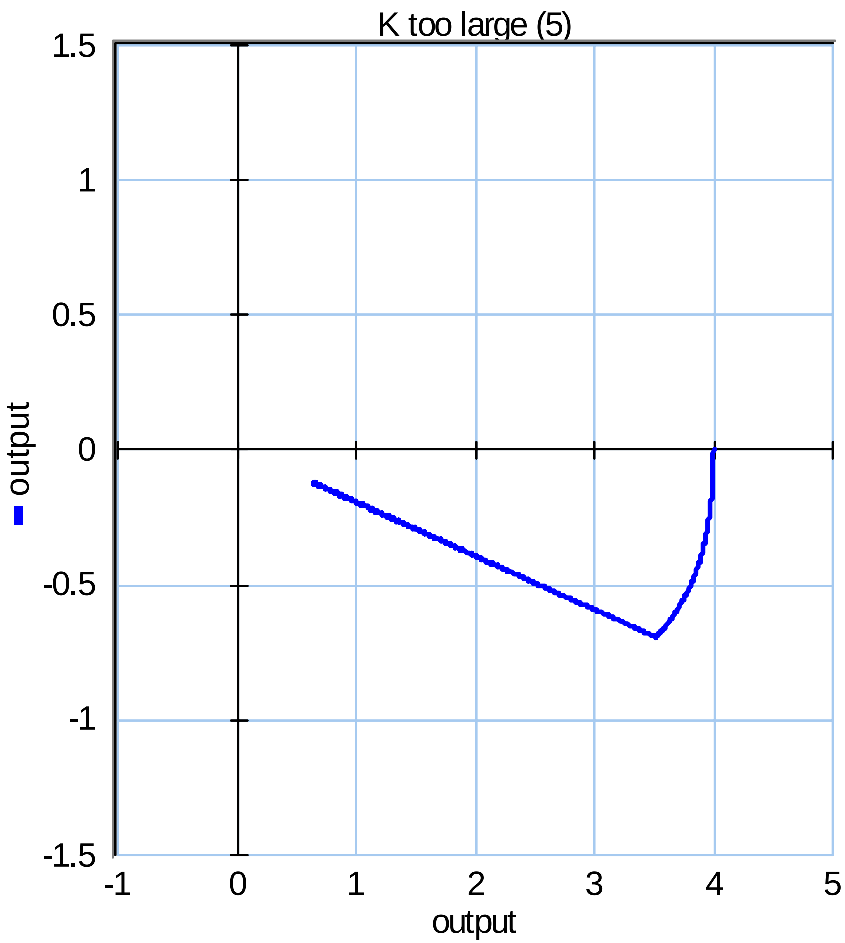
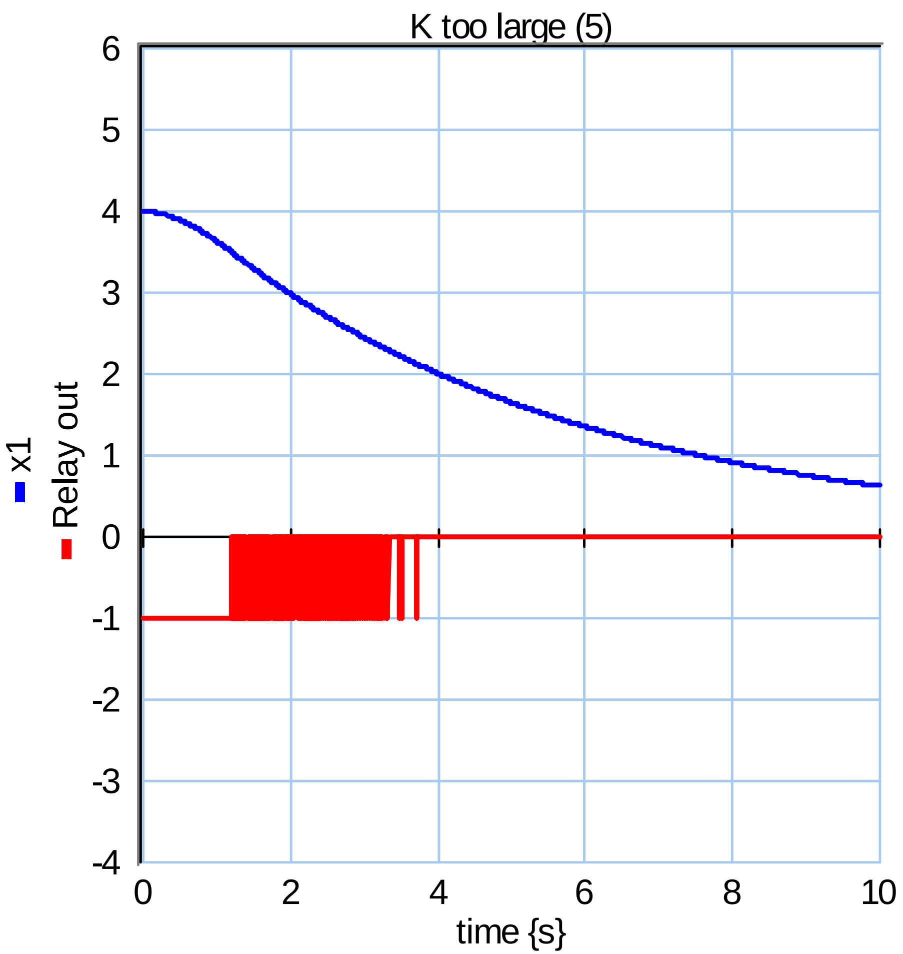
K_d too large





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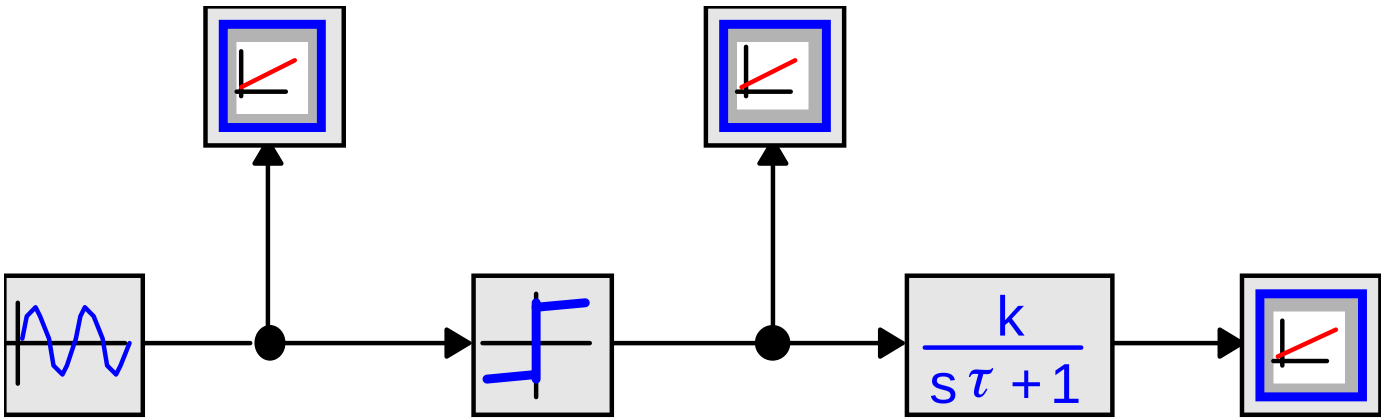
Describing functions

Describing functions



We consider systems
at the border of instability:
i.e. sinusoidal signals

Only first harmonic
of the signal,
because of low-pass
character of process

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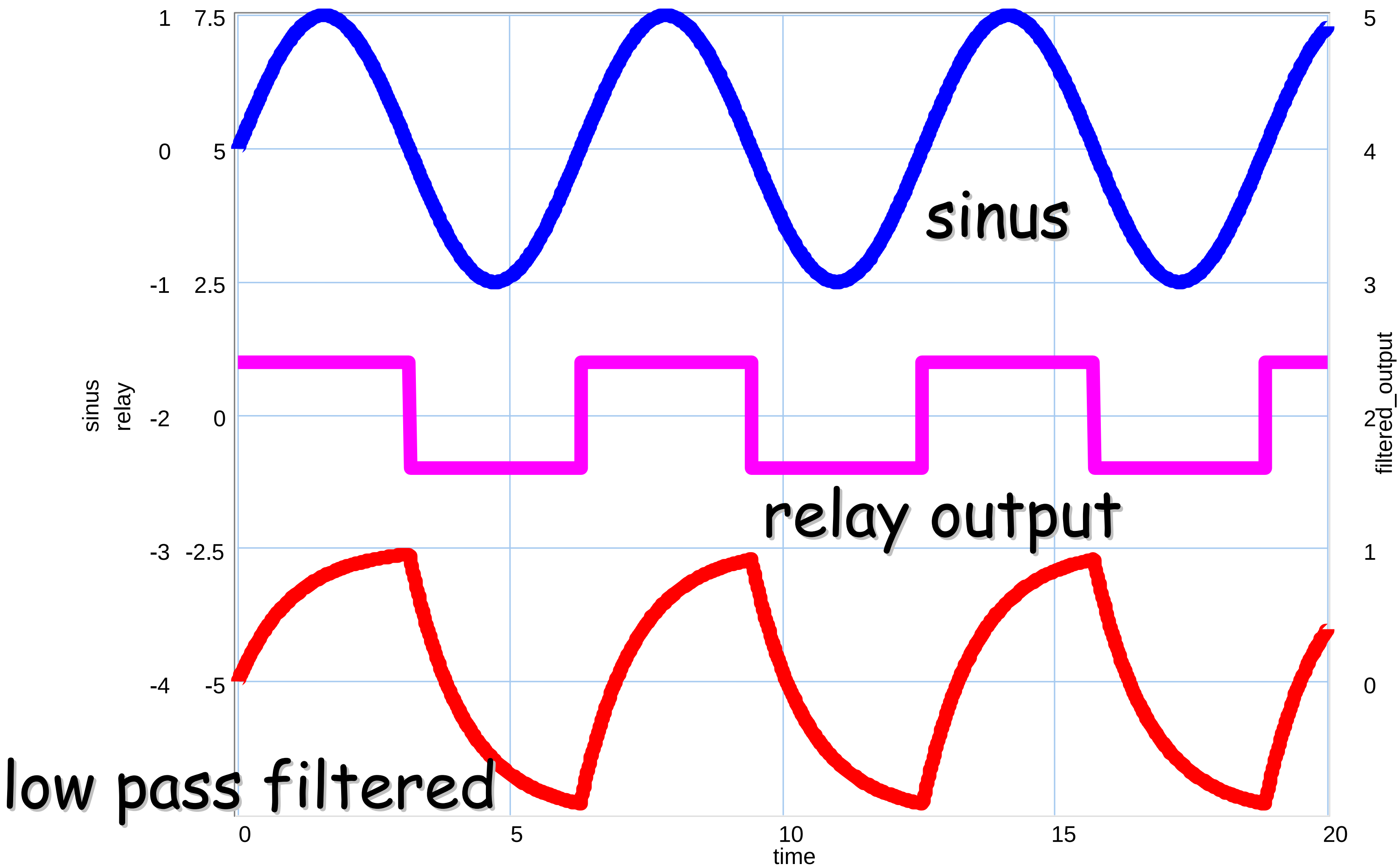
Signals



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Formulas

Consider a NL-element with output signal $y(t)$ and $x(t) = x_{max}\sin(\omega t)$ as input signal

Fourier series:

$$y(t) \approx y_1 \sin(\omega t + \varphi) = a_1 \cos(\omega t) + b_1 \sin(\omega t)$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} y \cos(\omega t)\, d(\omega t) \qquad y_1 = \sqrt{a_1^2 + b_1^2}$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} y \sin(\omega t)\, d(\omega t) \qquad \varphi = \arctan\left(\frac{a_1}{b_1}\right)$$

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Formulas

The describing function of the NL-element, N, is:

$$N = |N| e^{j\varphi} = \frac{y_1}{x_{max}} e^{j\varphi}$$

$$x_{max}\sin(\omega t) \longrightarrow \boxed{|N| e^{j\varphi}} \longrightarrow y_1 \sin(\omega t + \varphi)$$

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Describing function of relay

$$y(t) = M \quad \text{if } 0 < \omega t < \pi$$

$$y(t) = -M \quad \text{if } \pi < \omega t < 2\pi$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} y\cos(\omega t)\,d(\omega t) =$$

$$= \frac{1}{\pi}\int_0^{\pi} M\cos(\omega t)\,d(\omega t) + \frac{1}{\pi}\int_{\pi}^{2\pi} -M\cos(\omega t)\,d(\omega t)$$

$$a_1 = \frac{1}{\pi}\left(M\sin(\omega t)\right)\Big|_0^{\pi} + \frac{1}{\pi}\left(-M\sin(\omega t)\right)\Big|_{\pi}^{2\pi} = 0$$

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$\varphi = 0$

$$\varphi = \arctan\left(\frac{a_1}{b_1}\right)$$

For all radial symmetric, single valued non linearities, $a_1 = 0$.

This implies that $\varphi = 0$ (no phase lag)

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Describing function of relay

$$y(t) = M \text{ if } 0 < \omega t < \pi, \quad y(t) = -M \text{ if } \pi < \omega t < 2\pi$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} y \sin(\omega t)\, d(\omega t) =$$

$$= \frac{1}{\pi}\int_0^{\pi} M \sin(\omega t)\, d(\omega t) + \frac{1}{\pi}\int_{\pi}^{2\pi} -M \sin(\omega t)\, d(\omega t)$$

$$b_1 = \frac{1}{\pi}\left(-M\cos(\omega t)\right)\Big|_0^{\pi} + \frac{1}{\pi} - \left(-M\sin(\omega t)\right)\Big|_{\pi}^{2\pi} =$$

$$= \frac{1}{\pi}\left[-\left(-M - M\right)\right] + \frac{1}{\pi}\left[-\left(-M - M\right)\right] = \frac{4M}{\pi}$$

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Describing function of relay

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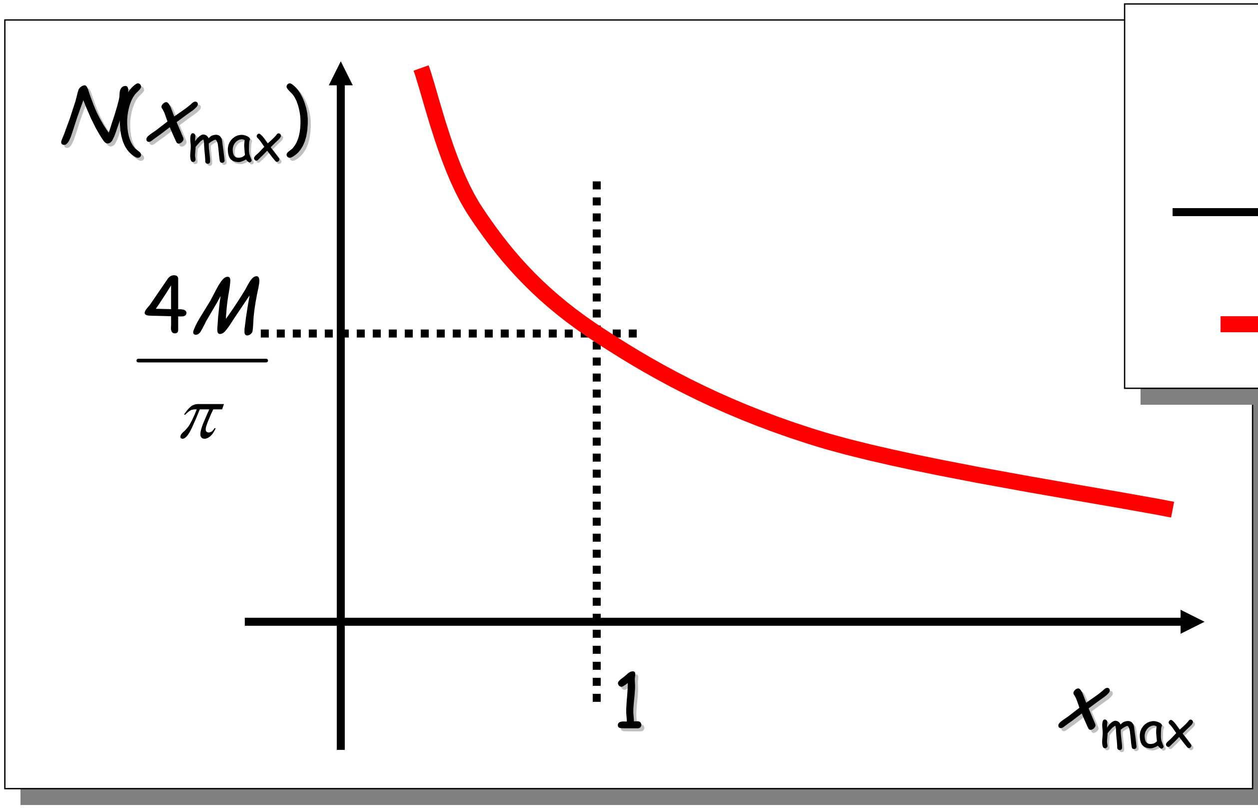
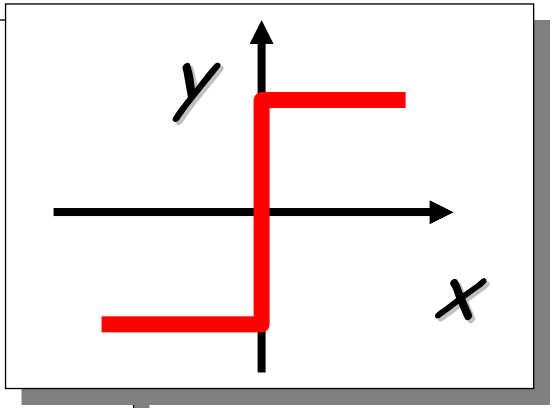
$$N = \frac{y_1}{x_{max}} = \frac{b_1}{x_{max}} = \frac{4M}{\pi}\frac{1}{x_{max}}$$





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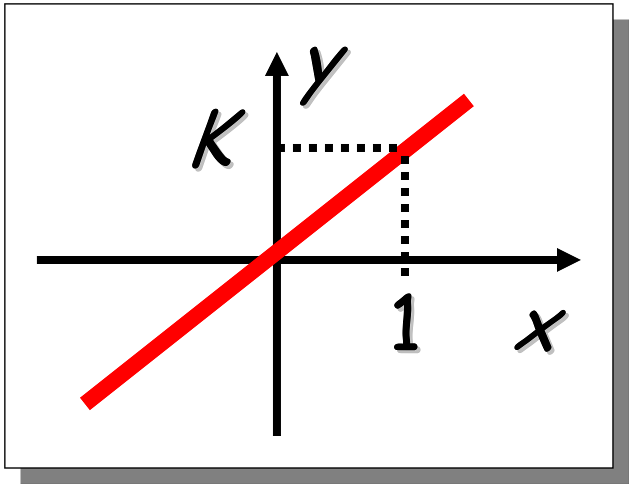
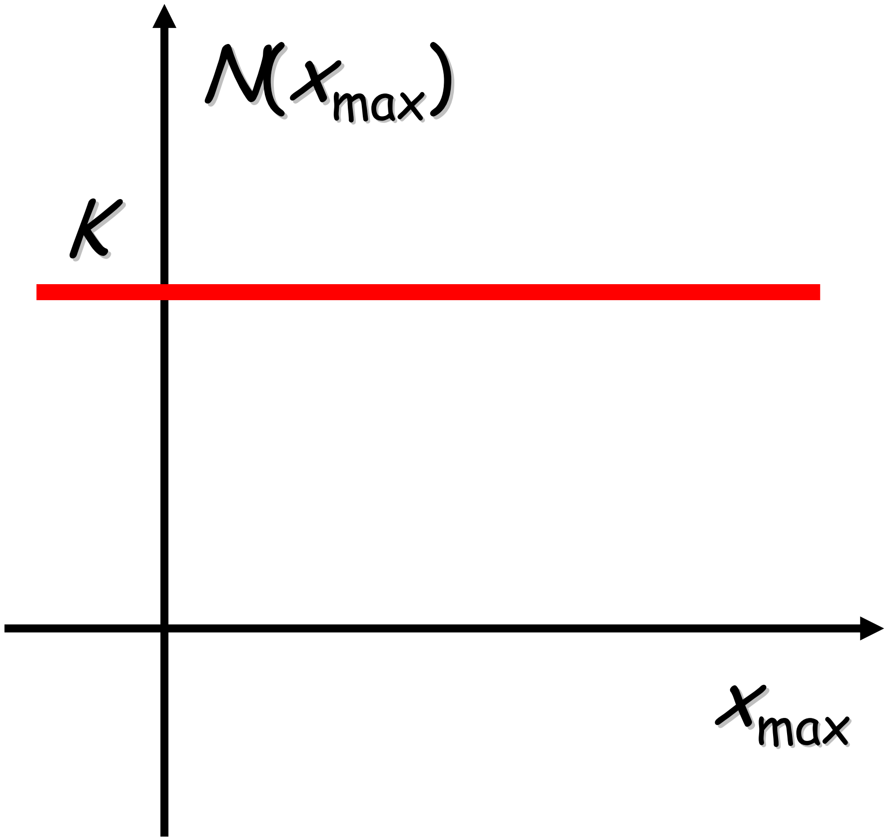
Describing function (linear gain)





DF is a kind of amplitude-dependent gain

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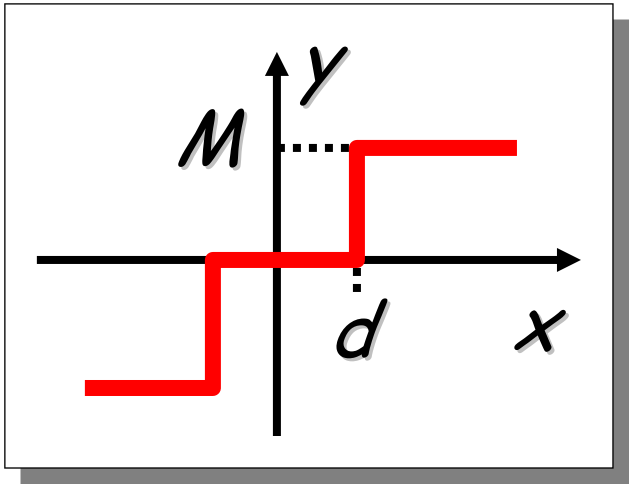
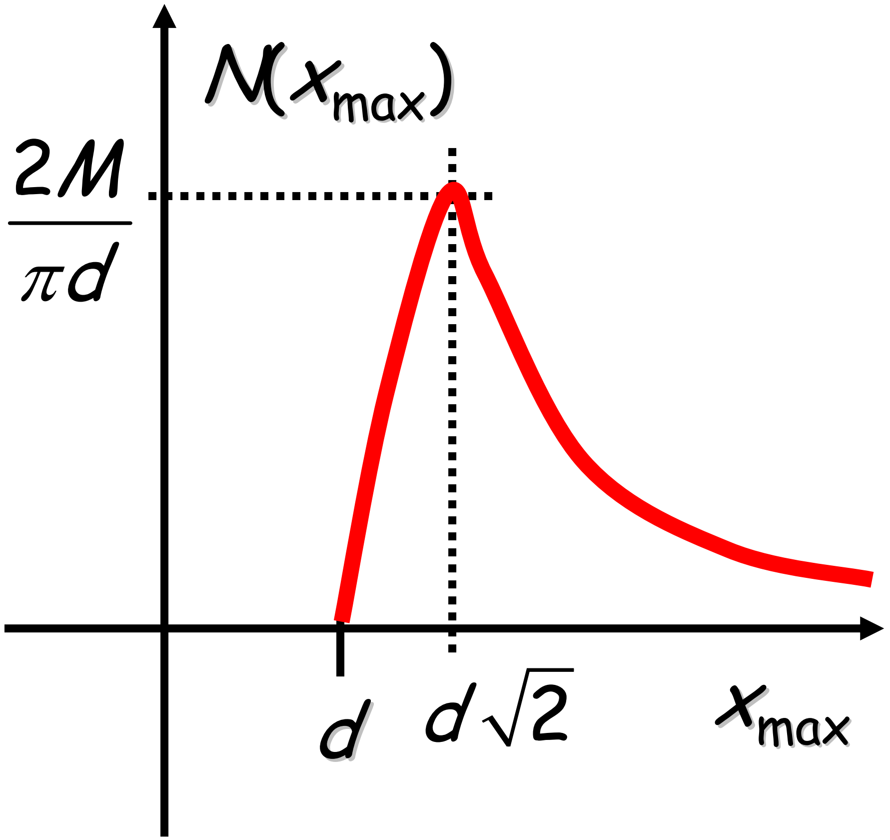
Describing function (relay with dead zone)





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Describing function (saturation)

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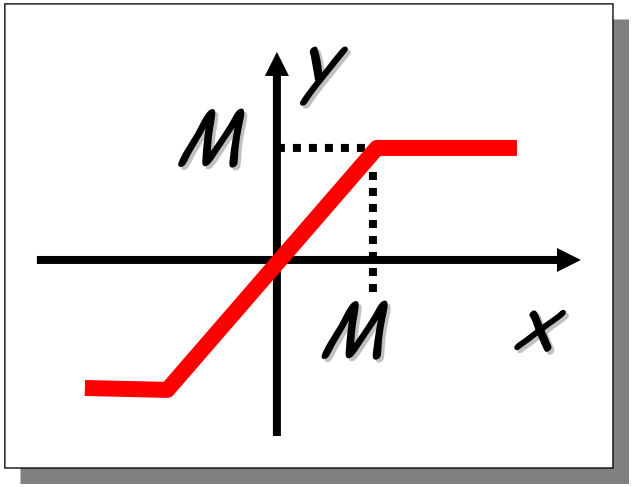

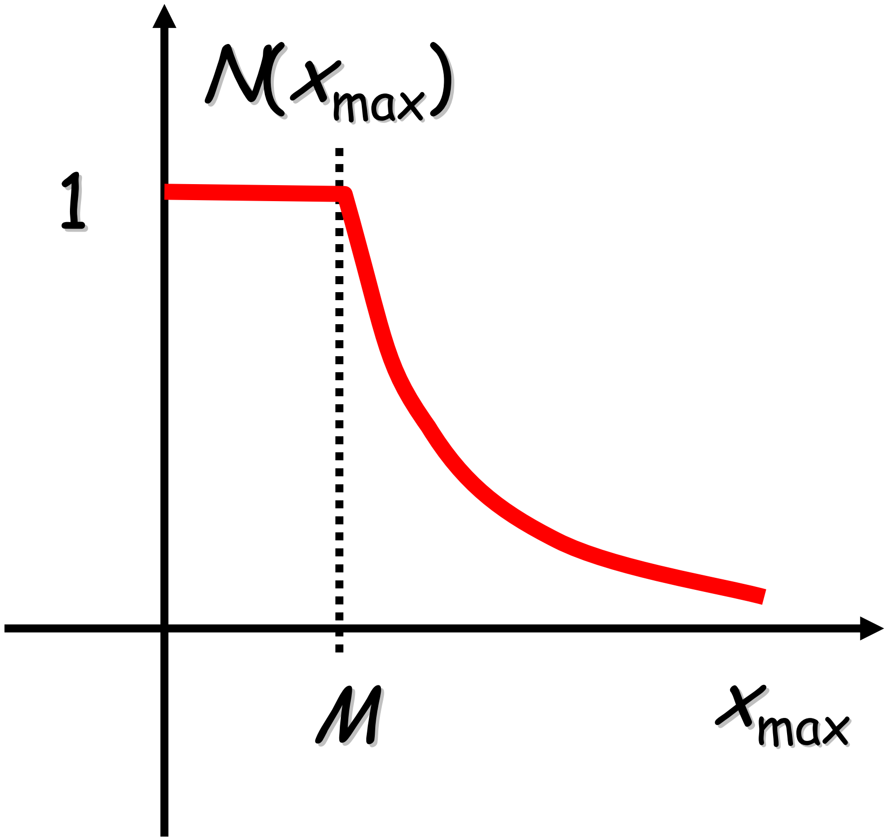

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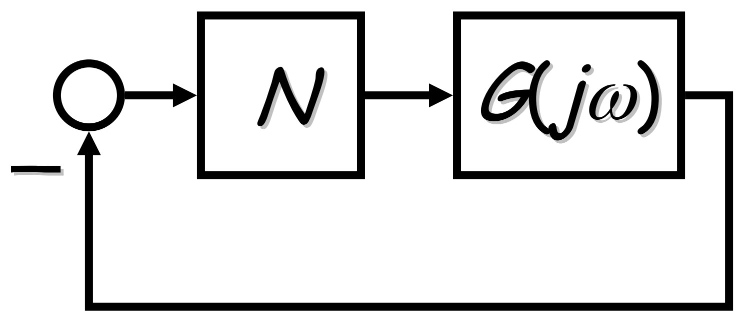
Use of describing function

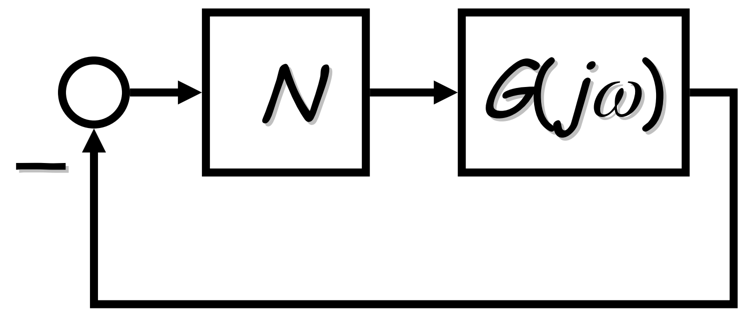
- Describing function should only be used for sinusoidal signals
  - systems on the border of instability



Stability analysis

Close loop system (characteristic equation)

$$1 + G(j\omega)N(x_{max}) = 0$$



$$G(j\omega)N(x_{max}) = -1$$

$$G(j\omega) = -\frac{1}{N(x_{max})}$$

Nyquist plot

Inverse of DF

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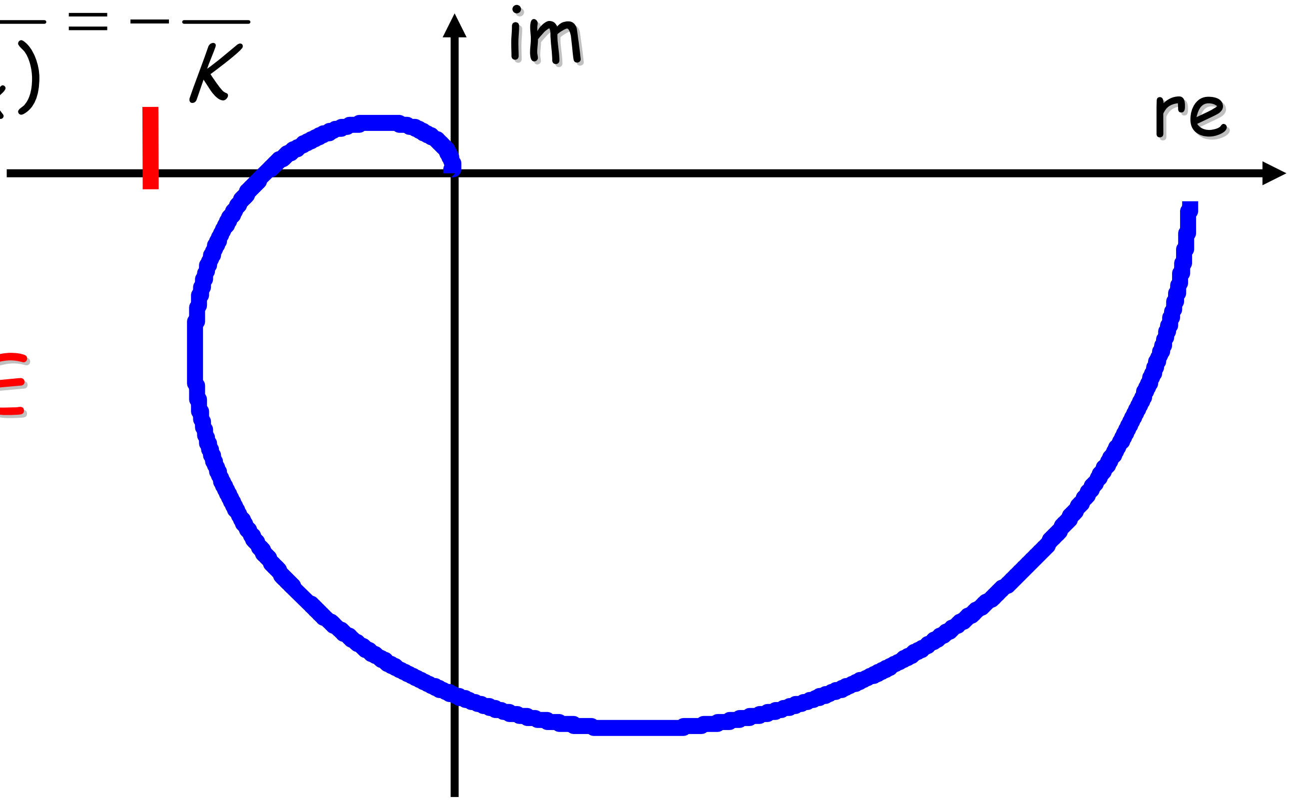
Example (gain)

$$-\frac{1}{N(x_{max})} = -\frac{1}{K}$$



STABLE

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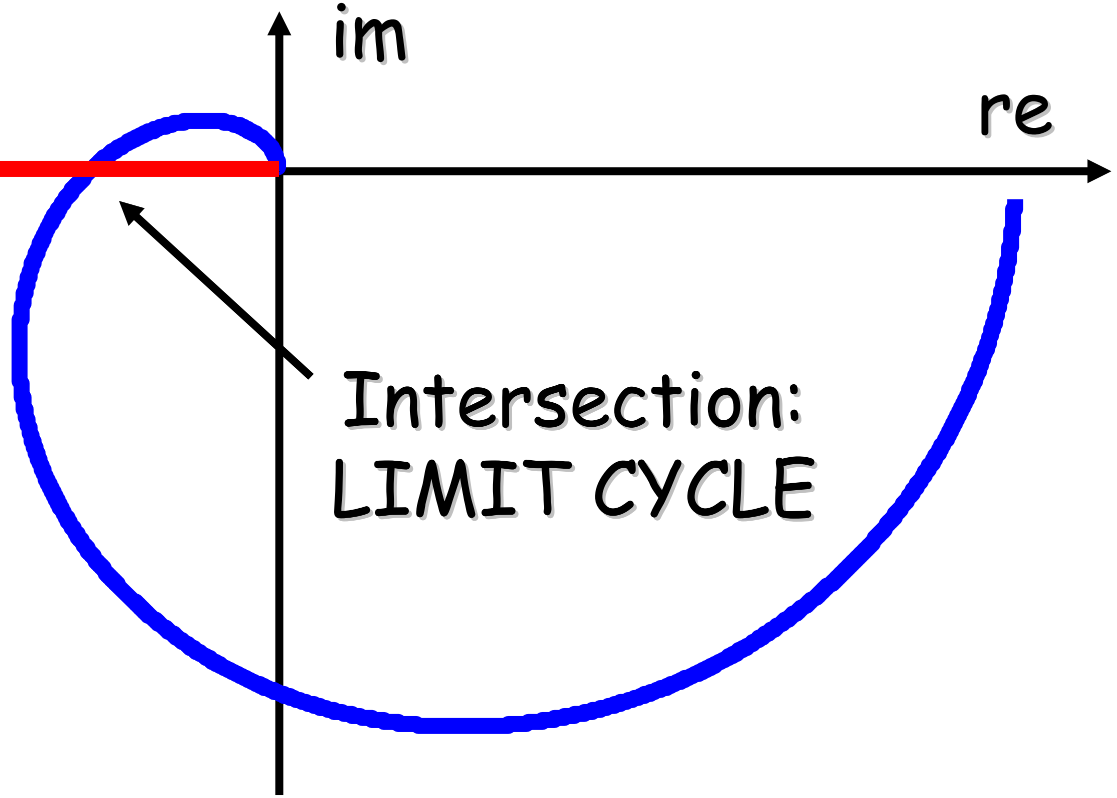
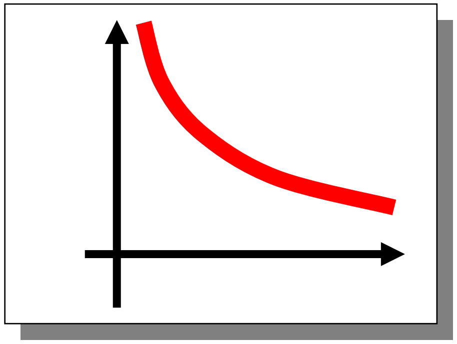
Example (relay)





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Limit cycle

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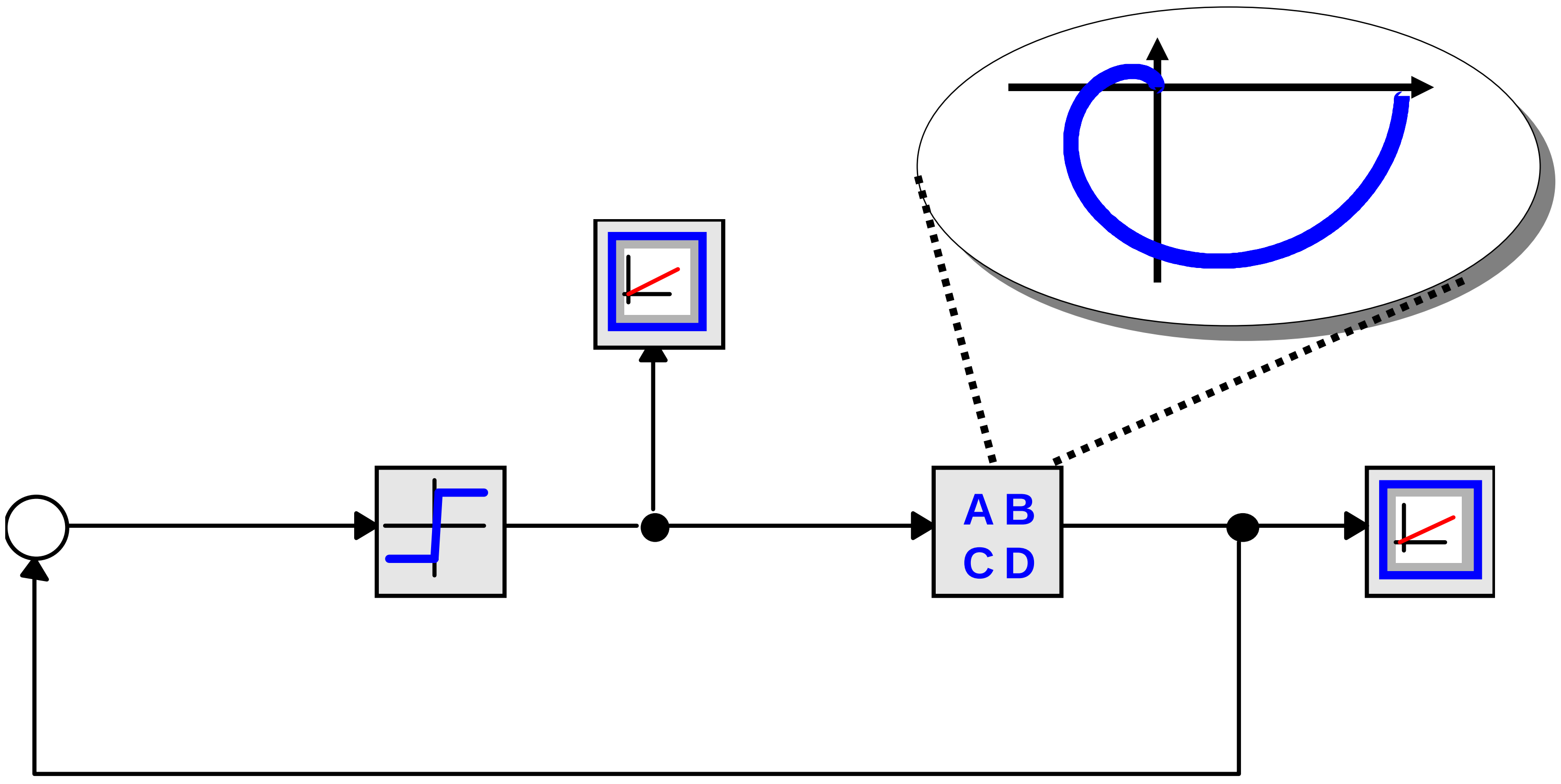

Limit_cycle_relay.em

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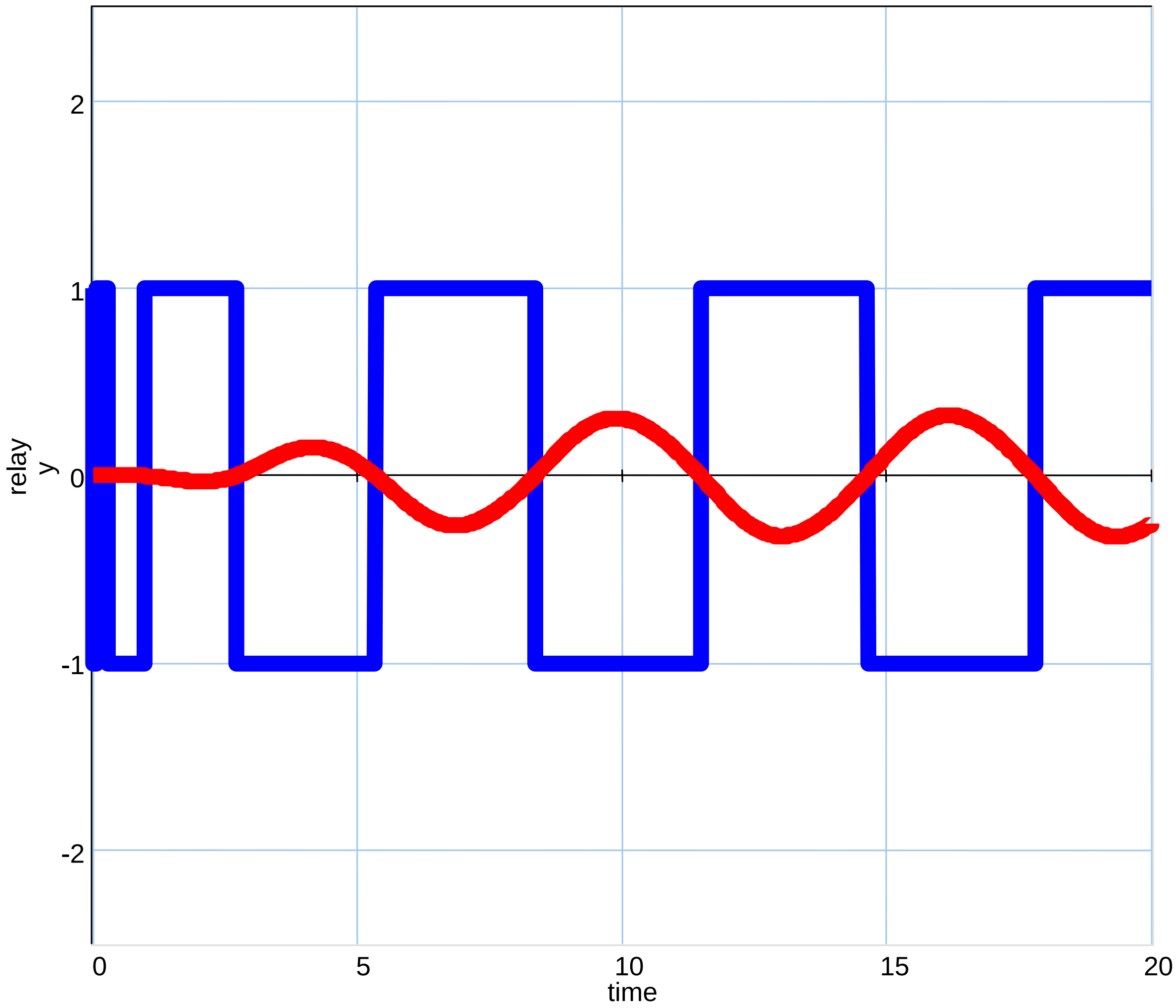
Response



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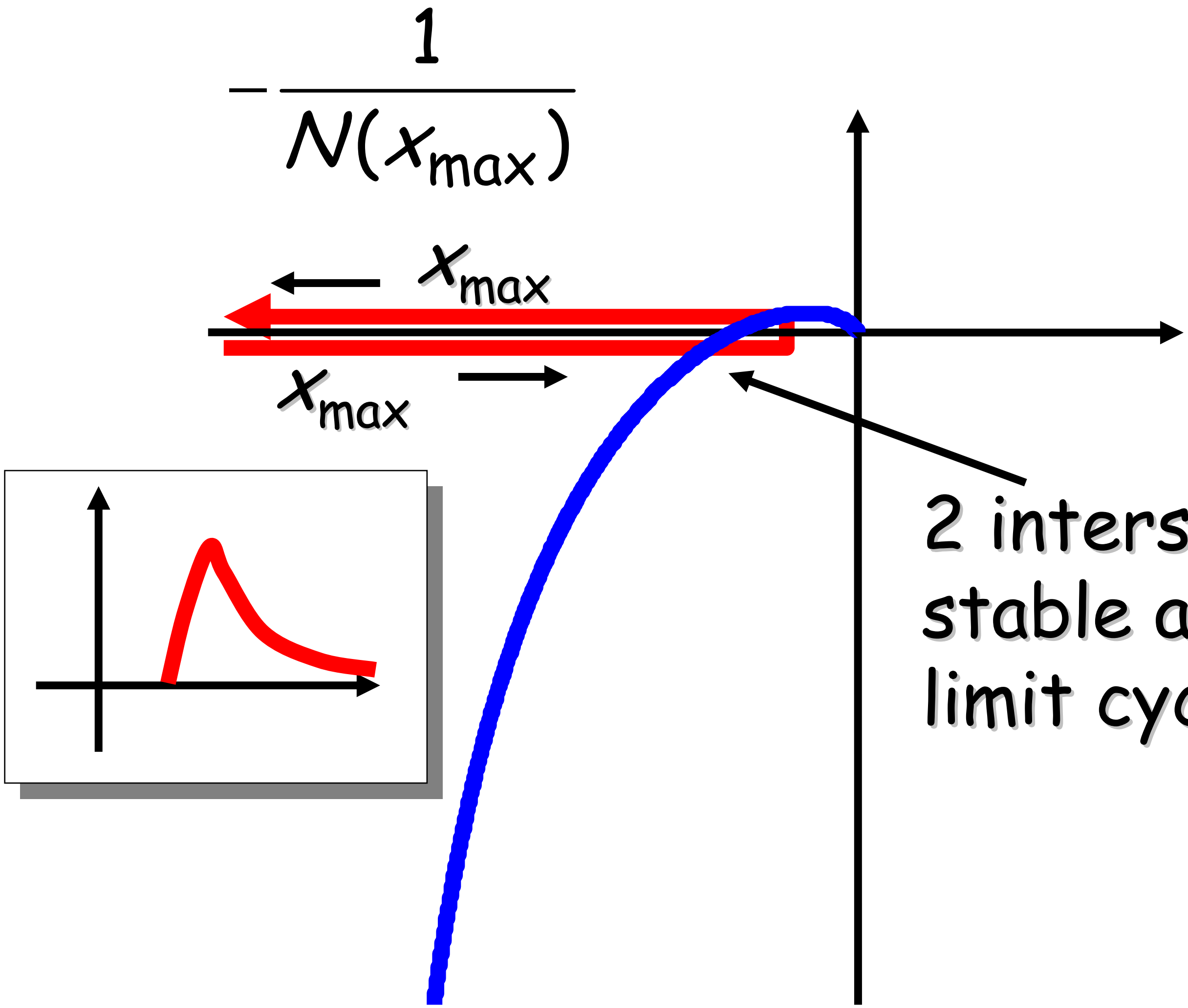
Example (relay + dead zone)



2 intersections:
stable and unstable
limit cycle

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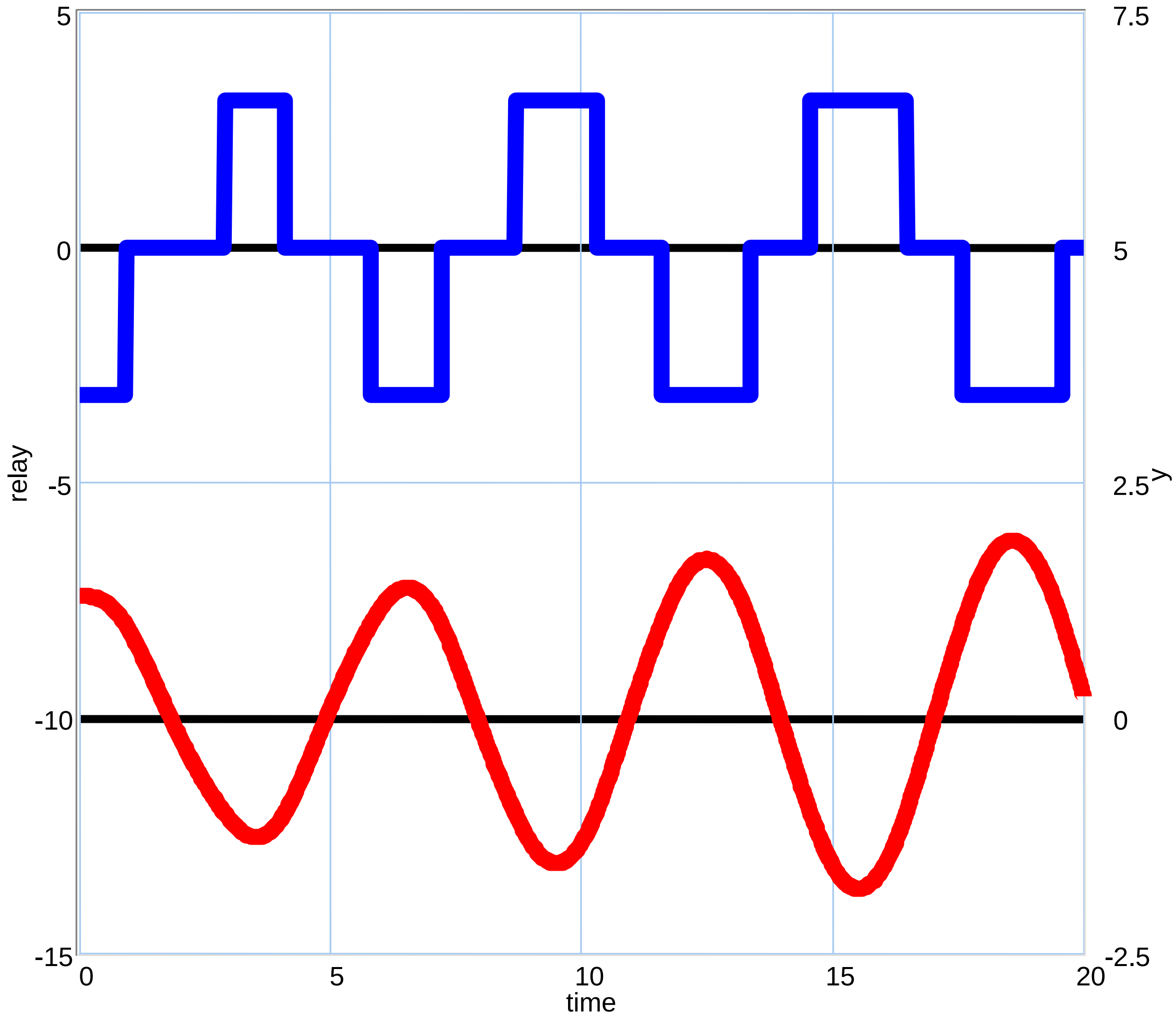
Response (1)

y(0) < d (=1)
no reaction

y(0) = 1.3



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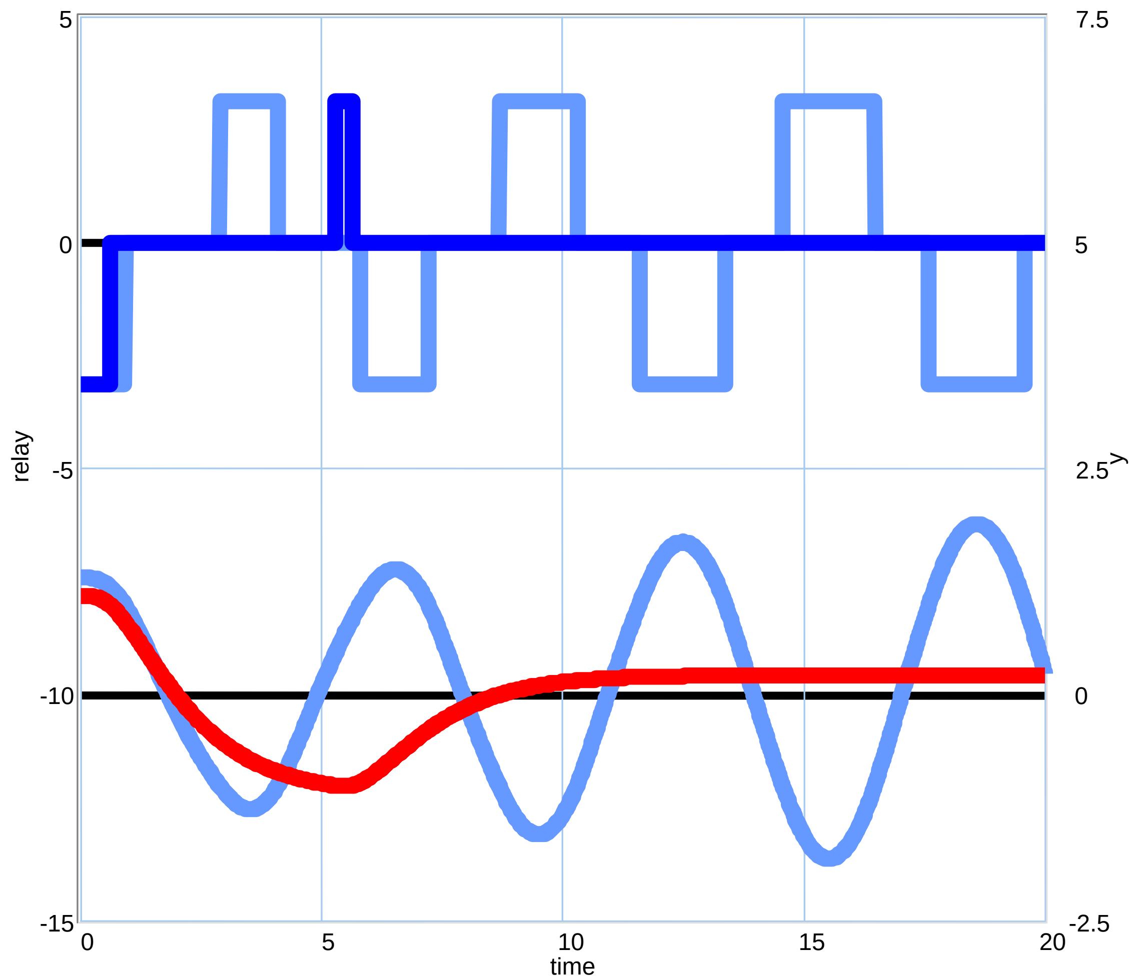
Response(2)

y(0) = 1.1



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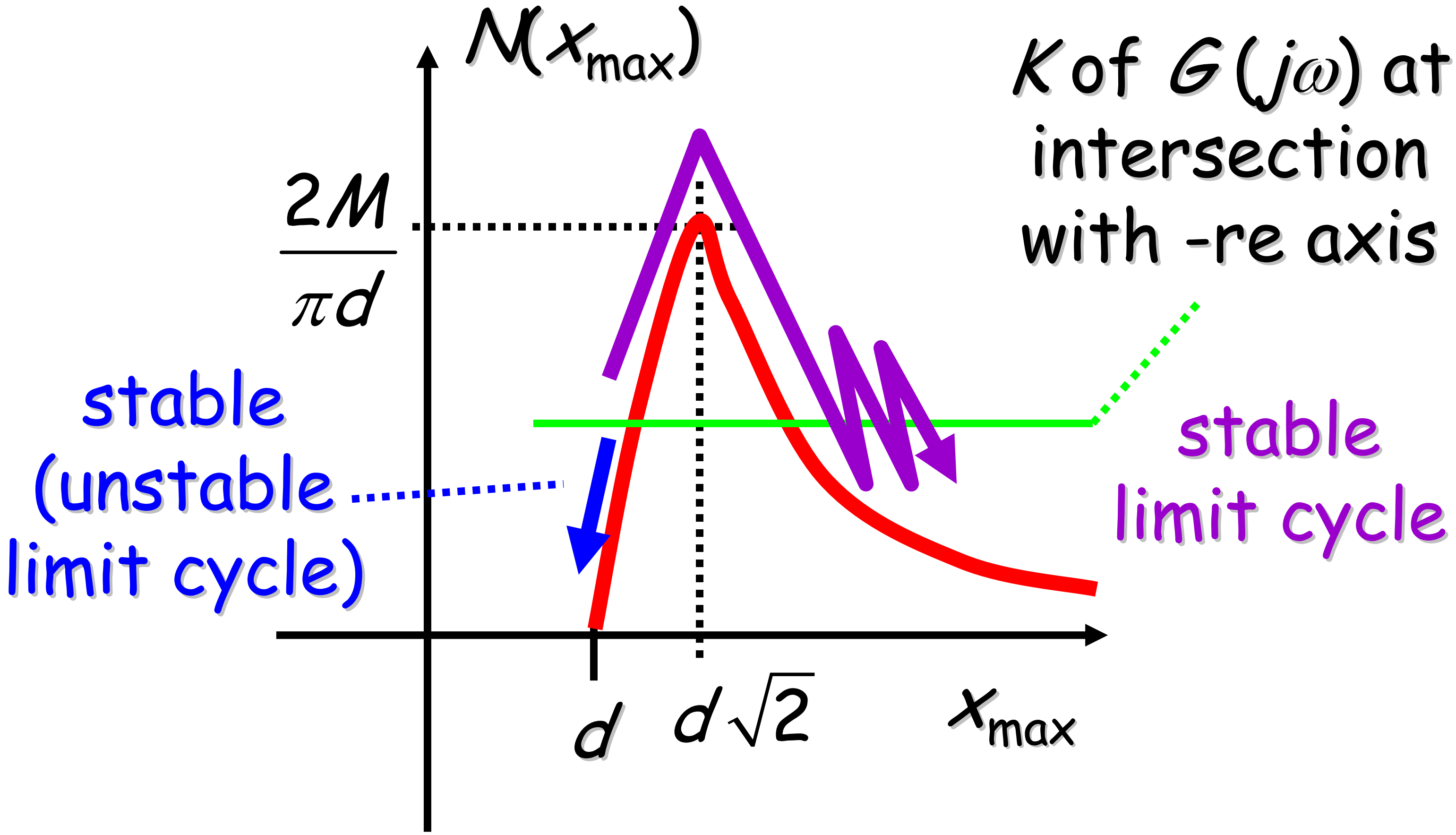
Explanation



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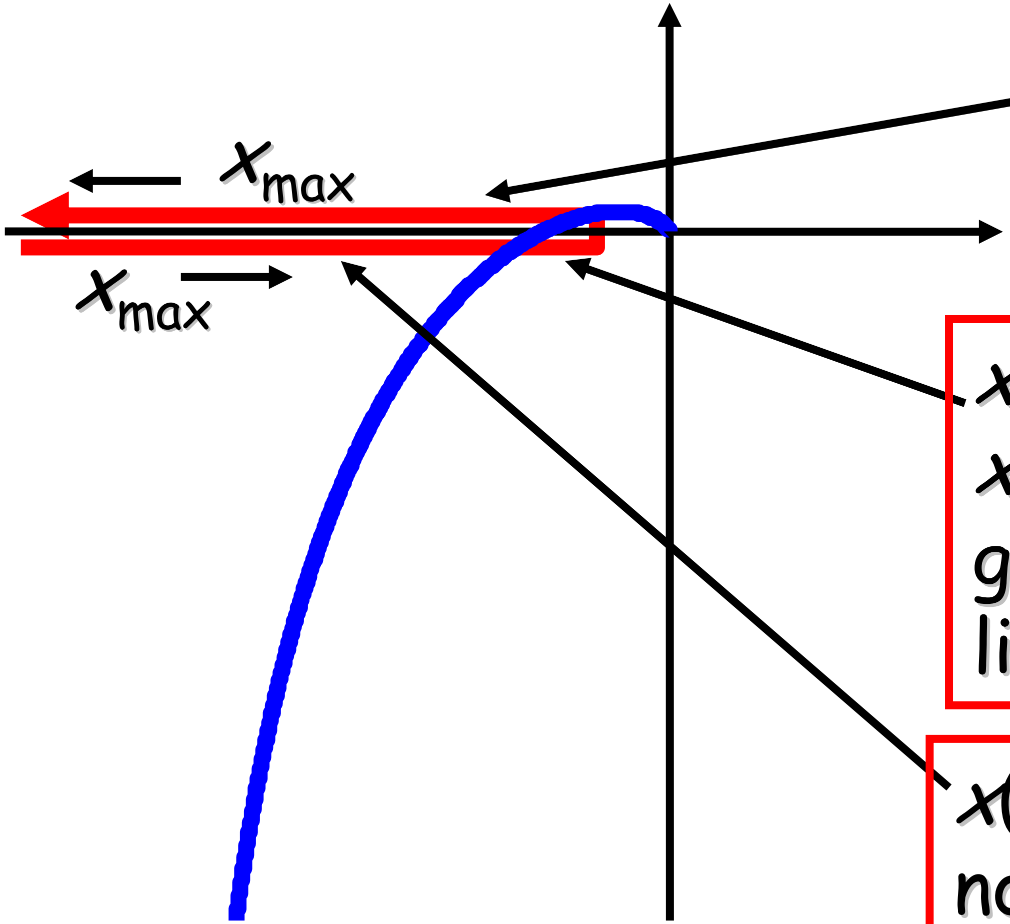
Example (relay + dead zone)



x(0) large
(stable limit cycle)

x(0) medium: 'unstable',
x_{max} increases,
goes to stable
limit cycle

x(0) small:
no reaction or unstable
limit cycle

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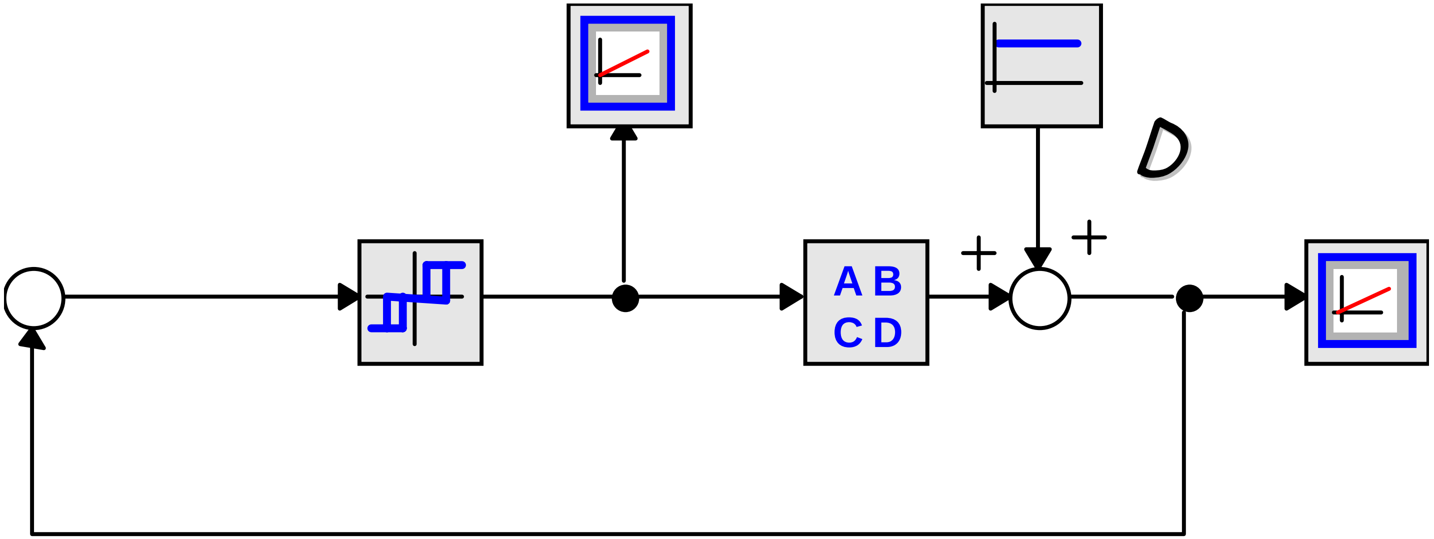

Limit_cycle_relay_dead_zone

Dead zone = 1

D = 0.8, 1.05, 1,1, 1.15

Conclusions

- Describing function gives an interpretation of non-linear systems
  - only at border of instability !
  - Only valid for more or less sinusoidal signals
  - (only first harmonic of Fourier series is taken into account)

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